The Embedded Graphs of a Knot and the Partial Duals of a Plane Graph

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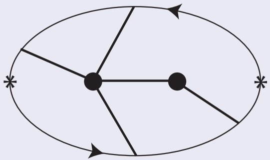
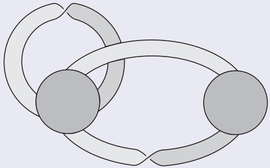
Ribbon graphs

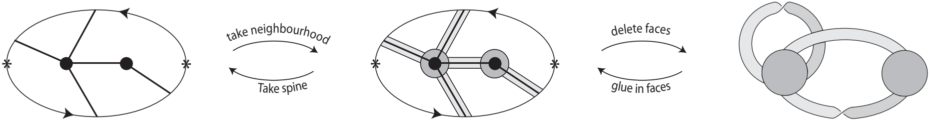
Ribbon graphs describe (cellularly) embedded graphs.

Cellularly embedded graph



Ribbon graph

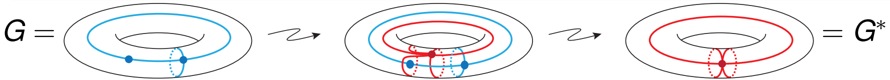




The geometric dual

The (geometric) dual G^* of a cellularly embedded graph G

- One vertex of G^* in each face of G.
- One edge of G^* whenever faces of G are adjacent.



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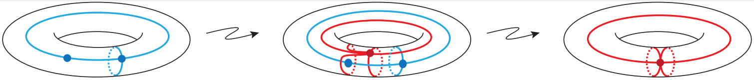
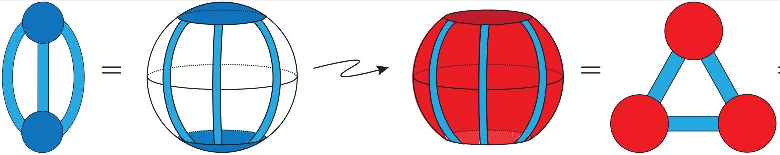
$G =$  $= G^*$

The (geometric) dual G^* of a ribbon graph G

- Fill in punctures of surface G with vertices of G^*,
- then delete vertices of G to get G^*.

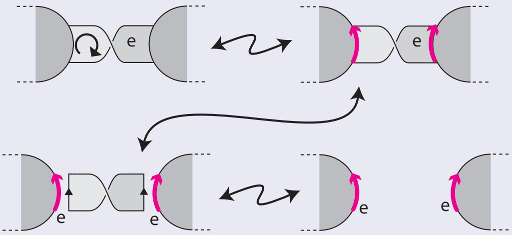
$G =$  $= G^*$

- Note: markings on G induce markings on G^*.

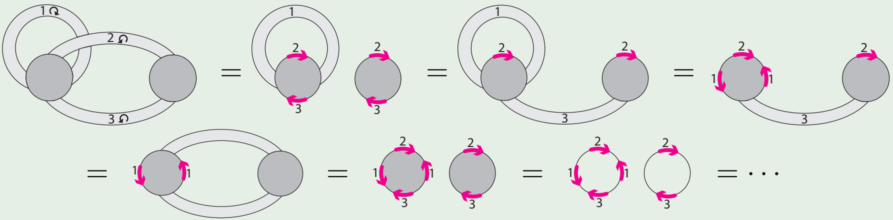
Arrow marked ribbon graphs

Edges can be described by pairs of coloured arrows on the boundary:

1. orient edge *e*
2. add arrows where *e* meets vertices
3. remove edge.



Example



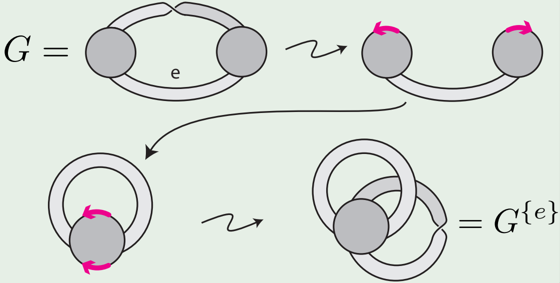
Partial duals

The **partial dual** G^A of G is obtained by forming the dual only at the edges in $A \subseteq E(G)$.

Definition: partial duals (S. Chmutov '07)

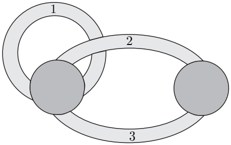
1. $A \subseteq E(G)$
2. Replace edges **not** in A by arrows.
3. Form geometric dual.
4. Add back edges.
5. Gives the partial dual G^A.

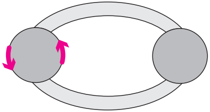
Example



Another example

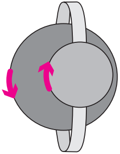
Forming G^A with $A = \{2, 3\}$.

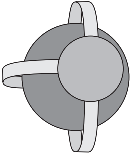
G= 

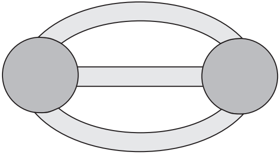
= 

1: given G and A

2: "hide" edges not in A



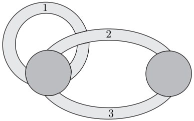
= 

= 

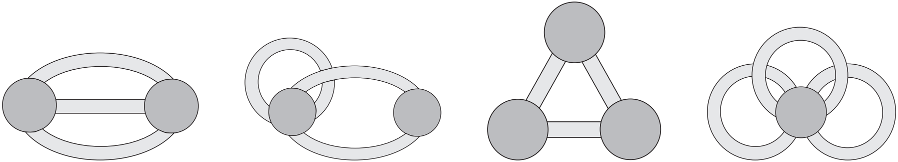
3: form the dual

4 & 5: add edge back to get G^A

The example continued...

G= 

has four partial duals (up to isomorphism):



- Observe that G and G^A can have very different graph theoretic and topological properties.

Some basic properties

- $G^{E(G)} = G^*$ and $G^{\emptyset} = G$.
- $(G^A)^A = G$. (In general, $(G^A)^B = G^{A \Delta B}$.)
- G orientable $\iff$ G^A orientable.

Many properties of duality extent to partial duality

- Topological Tutte polynomial is well behaved under partial duality.
- Unifies various connections between knot and graph polynomials.
- Admits algebraic characterization.
- Extends relations between duals and medial graphs to maps.
- Much remains to be explored!

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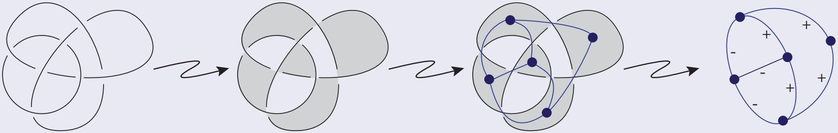
Advertisement.

Go to Jo Ellis-Monaghan's talk 10:30-10:55 Thursday to hear about our joint work on generalized duals, medial graphs and graph polynomials.

Tait graphs

There is a well known way to get a plane graph from a link diagram:



A little knot theory

Tait graphs

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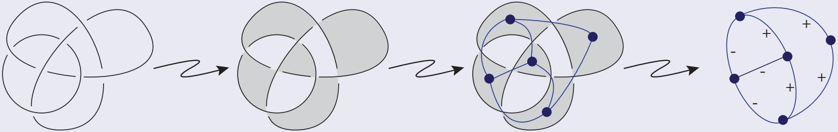
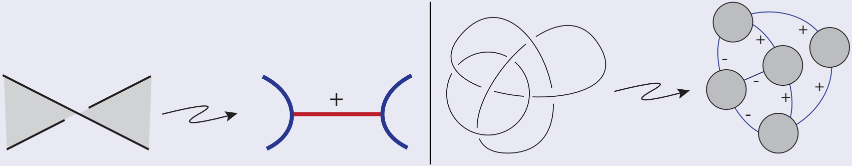


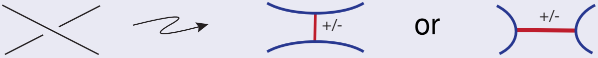


A little knot theory

The ribbon graphs of a link diagram (Dasbach, Futer, Kalfagianni, Lin & Stoltzfus ’06)

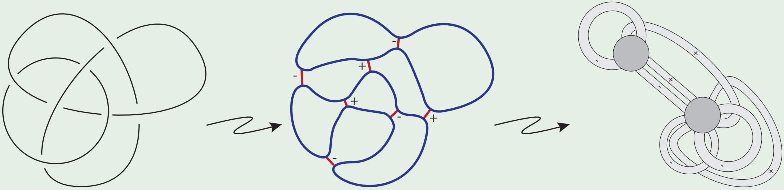
Associates $\leq 2^{\#crossings}$ ribbon graphs to a link diagram.

- Choose a (signed) smoothing at each crossing:



- Gives presentation of a ribbon graph:

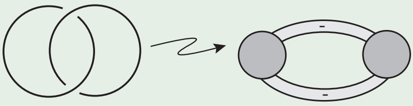
Example

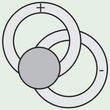


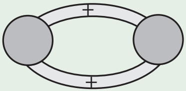
A question from knot theory

Example

The ribbon graphs of the Hopf link are:

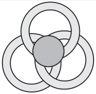


,


and


A fundamental question.

- Which ribbon graphs arise from link diagrams?

Not all of them. For example  doesn't.

A graph theoretic formulation.

- Which ribbon graphs are partial duals of plane graphs?

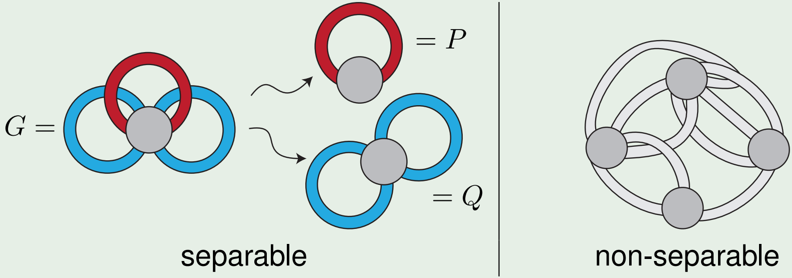
The answer has to do with the separability of a ribbon graph.

Separable ribbon graphs

Definition

- A **separation** of a ribbon graph G is a decomposition into two ribbon subgraphs P and Q which meet at exactly one vertex.
- The vertex where P and Q meet is a **separating vertex.**

Example



Separable ribbon graphs

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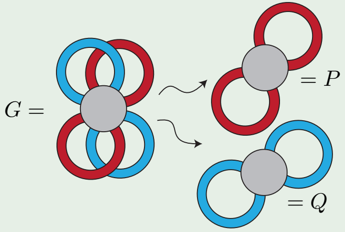
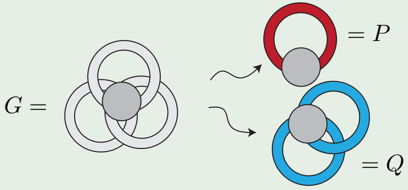
Example

We will be interested in separating ribbon graphs into plane graphs.



can be separated
into two plane graphs



can't be separated
into two plane graphs

1-decompositions into two plane graphs

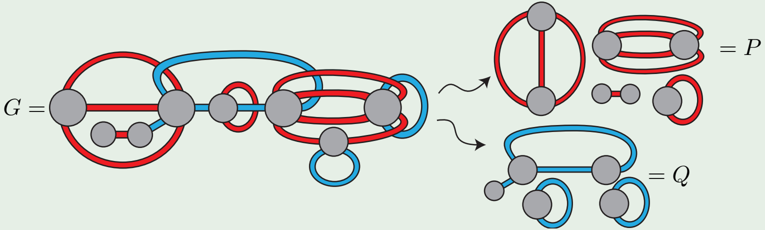
Definition

G has a 1**-decomposition into two graphs** if

1. G has a decomposition into two (not necessarily connected and possibly empty) ribbon subgraphs P and Q;
2. each vertex incident to edges in both P and Q is a separating vertex of the connected component in which it lies.

If P and Q are plane, the 1-decomposition is into **two plane graphs**.

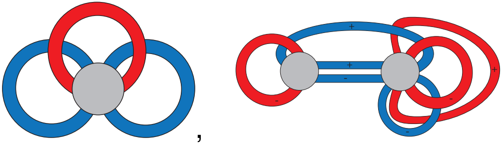
Example



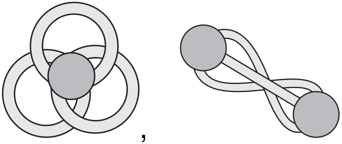
The Main Theorem

Theorem

An embedded graph G is a partial dual of a plane graph if and only if there exists a 1-decomposition of G into two plane graphs.



partial duals of plane graphs

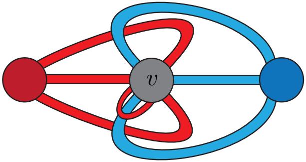


not p.ds of plane graphs

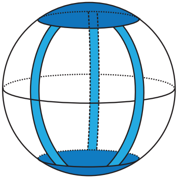
Theorem

Let G be an embedded graph and $A \subseteq E(G)$. Then G^A is a plane graph if and only if A defines a 1-decomposition of G into two plane graphs.

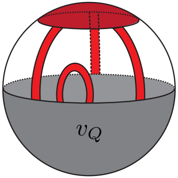
Starting with a 1-decomposition into two plane graphs



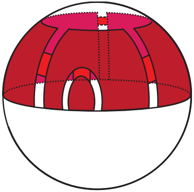
separate



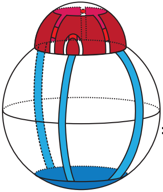
&



form P^*



join



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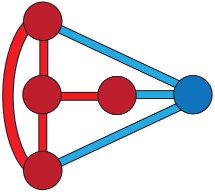
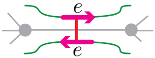
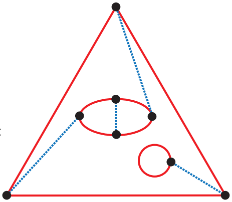
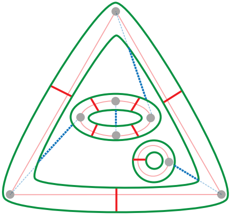
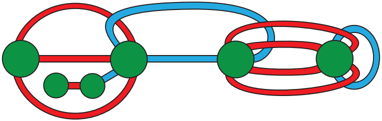


Idea of proof: “only if”

- Edges in A are red, edges not in A are blue.
- To construct partial dual G^A: 

If $e \in A$.
and 

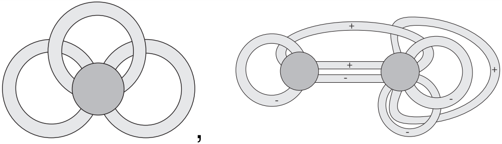
If $e \notin A$
- $G =$  presentation of G^A 
- Red/blue markers lie in different regions defining a 1-decomposition.
- $G^A =$ 

Back to link diagrams

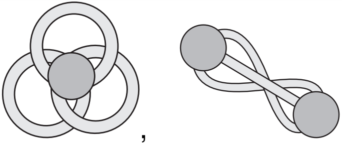
Recall we were motivated by understanding which ribbon graphs presented links.

Theorem

A connected (signed) embedded graph G represents a link diagram if and only if there exists a 1*-decomposition of G into two plane graphs.*



, present links



, don't present links

- All ribbon graphs of a link diagram are partial duals (of the Tait graphs).
- Can use separability result to classify all diagrams presented by the same ribbon graph.

References

- I. Moffatt, *Partial duals and the graphs of knots*.
- S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, `arXiv:0711.3490`.
- O. T. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, `arXiv:math.GT/0605571`.
- I. Moffatt, *A characterization of partially dual graphs*, `arXiv:0901.1868`.