The Embedded Graphs of a Knot and the Partial Duals of a Plane Graph

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SIAM Conference on Discrete Mathematics, 14\textsuperscript{th} June 2010
Ribbon graphs describe (cellularly) embedded graphs.
The geometric dual

The (geometric) dual $G^*$ of a cellularly embedded graph $G$

- One vertex of $G^*$ in each face of $G$.
- One edge of $G^*$ whenever faces of $G$ are adjacent.
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The (geometric) dual $G^*$ of a ribbon graph $G$

- Fill in punctures of surface $G$ with vertices of $G^*$,
- then delete vertices of $G$ to get $G^*$.

Note: markings on $G$ induce markings on $G^*$. 
Arrow marked ribbon graphs

Edges can be described by pairs of coloured arrows on the boundary:

1. orient edge $e$
2. add arrows where $e$ meets vertices
3. remove edge.

Example

\[
\begin{align*}
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \\
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \\
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} & \quad \begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \\
\end{align*}
\]
The **partial dual** \( G^A \) of \( G \) is obtained by forming the dual only at the edges in \( A \subseteq E(G) \).

**Definition: partial duals (S. Chmutov ’07)**

1. \( A \subseteq E(G) \)
2. Replace edges **not** in \( A \) by arrows.
3. Form geometric dual.
4. Add back edges.
5. Gives the partial dual \( G^A \).

**Example**

\[
G = \begin{array}{ccc}
A & \sim & B \\
\end{array}
\quad \Rightarrow \quad \begin{array}{ccc}
A & \sim & B \\
\end{array}
\]

\( G \{e\} \)

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Partial Duals of a Plane Graph

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Another example

Forming $G^A$ with $A = \{2, 3\}$.

1: given $G$ and $A$

2: “hide” edges not in $A$

3: form the dual

4 & 5: add edge back to get $G^A$
The example continued...

G= has four partial duals (up to isomorphism):

- Observe that $G$ and $G^A$ can have very different graph theoretic and topological properties.
Some basic properties

- \( G^E(G) = G^* \) and \( G^\emptyset = G \).
- \( (G^A)^A = G \). (In general, \( (G^A)^B = G^{A \Delta B} \).)
- \( G \) orientable ⇔ \( G^A \) orientable.

Many properties of duality extent to partial duality

- Topological Tutte polynomial is well behaved under partial duality.
- Unifies various connections between knot and graph polynomials.
- Admits algebraic characterization.
- Extends relations between duals and medial graphs to maps.
- Much remains to be explored!
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Advertisement.

Go to Jo Ellis-Monaghan’s talk **10:30-10:55 Thursday** to hear about our joint work on generalized duals, medial graphs and graph polynomials.
There is a well known way to get a plane graph from a link diagram:

Tait graphs
Tait graphs

There is a well known way to get a plane graph from a link diagram:

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Partial Duals of a Plane Graph

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The ribbon graphs of a link diagram (Dasbach, Futer, Kalfagianni, Lin & Stoltzfus ’06)

Associates $\leq 2^\#\text{crossings}$ ribbon graphs to a link diagram.

- Choose a (signed) smoothing at each crossing:
  - 
  - 
  - 
  - 

- Gives presentation of a ribbon graph:
  - 
  - 

Example
A question from knot theory

Example

The ribbon graphs of the Hopf link are:

A fundamental question.

- Which ribbon graphs arise from link diagrams?

Not all of them. For example, \( \) doesn’t.

A graph theoretic formulation.

- Which ribbon graphs are partial duals of plane graphs?

The answer has to do with the separability of a ribbon graph.
**Separable ribbon graphs**

**Definition**
- A **separation** of a ribbon graph $G$ is a decomposition into two ribbon subgraphs $P$ and $Q$ which meet at exactly one vertex.
- The vertex where $P$ and $Q$ meet is a **separating vertex**.

**Example**

$G = \begin{array}{c}
\text{separable} \\
\end{array}$

$= P$

$= Q$

$= Q$

non-separable

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Separable ribbon graphs

**Definition**
- A *separation* of a ribbon graph $G$ is a decomposition into two ribbon subgraphs $P$ and $Q$ which meet at exactly one vertex.
- The vertex where $P$ and $Q$ meet is a *separating vertex*.

**Example**
We will be interested in separating ribbon graphs into plane graphs.

- $G = \begin{array}{c}
\begin{array}{c}
\text{can be separated into two plane graphs}
\end{array}
\end{array}$

- $G = \begin{array}{c}
\begin{array}{c}
\text{can’t be separated into two plane graphs}
\end{array}
\end{array}$
**Definition**

A graph \( G \) has a **1-decomposition into two graphs** if:

1. \( G \) has a decomposition into two (not necessarily connected and possibly empty) ribbon subgraphs \( P \) and \( Q \);
2. each vertex incident to edges in both \( P \) and \( Q \) is a separating vertex of the connected component in which it lies.

If \( P \) and \( Q \) are plane, the 1-decomposition is into **two plane graphs**.

**Example**

Graph \( G \) is decomposed into subgraphs \( P \) and \( Q \).
The Main Theorem

**Theorem**

An embedded graph $G$ is a partial dual of a plane graph if and only if there exists a $1$-decomposition of $G$ into two plane graphs.

![Diagram of partial duals and non-duals of plane graphs]

**Theorem**

Let $G$ be an embedded graph and $A \subseteq E(G)$. Then $G^A$ is a plane graph if and only if $A$ defines a $1$-decomposition of $G$ into two plane graphs.
Idea of proof: “if”

Starting with a 1-decomposition into two plane graphs

form $P^*$  join  $= vQ$
Idea of proof: “only if”

- Edges in $A$ are red, edges not in $A$ are blue.
- To construct partial dual $G^A$:
  - If $e \in A$.
  - If $e \notin A$.

$G = \ldots$ presentation of $G^A$

Red/blue markers lie in different regions defining a 1-decomposition.

$G^A = \ldots$
Recall we were motivated by understanding which ribbon graphs presented links.

**Theorem**

A connected (signed) embedded graph $G$ represents a link diagram if and only if there exists a 1-decomposition of $G$ into two plane graphs.

All ribbon graphs of a link diagram are partial duals (of the Tait graphs).

Can use separability result to classify all diagrams presented by the same ribbon graph.
I. Moffatt, *Partial duals and the graphs of knots*.

