What is a ribbon graph

- Informally, a ribbon graph is a "topological graph," with
  * discs ♦️ for vertices
  * ribbons 🎨 for edges

* Eq.

- I'll skip a formal definition since a picture is more effective.

- We can read off standard graph parameters:
  \( v(G) = \# \text{ vertices} \)
  \( e(G) = \# \text{ edges} \)
  \( b(G) = \# \text{ components} \)

- But also some topological parameters
  \( f(G) = \# \text{ boundary components} \)
  \( g(G) = \text{ genus} \)
  \( \Delta(G) = \begin{cases} 2 \times g(G) & \text{if orient} \\ g(G) & \text{if non-orient} \end{cases} \)

- Euler's Formula:
  \( v(G) - e(G) + f(G) = 2 \times b(G) - \Delta(G) \)

- Ribbon graphs describe exactly cellularly embedded graphs

- Ribbon graphs are equivalent to describe equiv. cell emb. graphs

- i.e., \( G \cong G' \) if homeo, taking \( G \) to \( G' \) that preserves
  - edges
  - vertices
  - cyclic order
  - orientation if \( G \) orientable

- Warning:
  - Ribbon graphs are not embedded in \( \mathbb{R}^3 \)
  - No concept of a non-loop edge being "twisted"
Arrow presentations

- A combinatorial description of a ribbon graph.
- Set of circles
- Pairs of directed arrows
Deletion

- Edge deletion is defined by removing an edge $e$ of $G$.
  \[ G \setminus e \text{ (or } G \setminus v \text{)} \]
  - Order of deletion does not matter so $A = \{ e \}$, and $G_A = G \setminus e$. 
- Vertex deletion: remove $v$ and its incident edges.
  \[ G \setminus v \]

- $H$ is a ribbon subgraph of $G$ if it can be obtained by deleting vertices and edges of $G$.
- $H$ is spanning if $V(G) = V(H)$ (so $H = G \setminus A$)

Example:

Ribbon graphs are obviously closed under deletion

But: Surfaces change when viewing as cell emb. graphs:

\[ \text{torus} \]

Remark:
Closure under deletion is a main reason for working with ribbon graphs:

- cell emb.
- not cell emb.

so cell emb. graphs are not (naturally) closed under deletion.

Exercise:
Add presentations

Arrow presentations

How should we define contraction?
- Three routes to the same vertex
- Not a loop
- Not a null

identity evvuv into a new vertex

Obvious case: care is needed

Cell ends graphs
- For the quotient space resolve any point points
- Each end point
- Each boundary point duplicate of uvw

Nulsen graphs
- Attatch a cicle to each boundary point
- Label uvw

It e's a loop all is fine!
Minors

Graph minors

$H$ minor of $G$ if obtained by vertex deletion, edge deletion, edge contraction.

Example

- $G$ tree $\iff$ no $O$-minor
- $G$ plane $\iff$ no $K_5$, $K_{3,3}$-minor

A family $\mathcal{F}$ is minor-closed if $G \in \mathcal{F} \Rightarrow \text{all minors of } G \in \mathcal{F}$.

Robertson–Seymour

- Every minor-closed family is characterised by a finite set of excluded minors.
- In any infinite collection of graphs, one is a minor of another.

Example

- $G$ embeddable in $\mathbb{R}^2$ $\Rightarrow$ no minor is a $K_4$ or $K_{3,3}$
- $G$ embeddable in $\Sigma$ $\Rightarrow$ no minor is a finite list.

Big problem: find excluded minor characterisations of various families.

Ribbon graph minors

- Ribbon graphs with rigidity, del. + cont.

Example

- $\mathcal{R} = (\mathcal{R}_5, \mathcal{R}_6)$
- Every $\mathcal{R}$ is $\mathcal{R}_5$ has $\mathcal{R}_6$ as a minor.

(But $\mathcal{R}_6$)

(Note that without contraction of loops, this would be an infinite antichain.)

Want to find excluded minor characterisations of various families.

Then $G$ orientable $\iff$ no $O$-minor.

Proof:

- Non-orient $\Rightarrow$ $\exists$ simple ribbon roving cycle
- $\Rightarrow$ $\exists$ cycle has a modulus band
- $\Rightarrow$ delete every diagonal in cycle, contract all but one edge
- $G$ orient $\Rightarrow$ all minors orient $\Rightarrow$ no $O$-minor.
proof

- $\chi(G) \leq \chi'(G)$
- so if $\chi(G) = 1$, $\chi'(G) \geq 1$.

Converse

- Contract spanning trees of $G$ to get union of 1 vertex.
- If an edge touches more than 1 $\chi$-crt, delete it.
- Reduces $\chi$-crts
- Does not change genus.

enough to show in hurewicz

- bonding opt no isolated vertices.
- there is always an edge we can delete that drops
- $\chi$ by 2 if orient
- $\chi$ by 1 if non-orient.

we have seen orient case.

so assume non-orient.

- \[
\begin{array}{c}
\text{slide} \\
\text{slide} \\
\text{slide}
\end{array}
\]

- property true before and after slide.
- normal form.

Note: $G-e$ could be orient or non-orient.

* not true if more than 1 $\chi$-crt:
\[
\chi(G-e) = 1
\]

eg
\[
\chi = 1
\]