POLYNOMIALS

Tutte polynomial

- e a <u>bridge</u> $\Leftrightarrow k(G\setminus e) > k(G)$
- e a <u>loop</u> $\Leftrightarrow$ incident to exactly one vertex

- <u>Tutte polynomial</u> is a polynomial valued graph invariant

- <u>Tutte polynomial</u> (1954)

$T(G; x, y) \in \mathbb{Z}[x, y]$

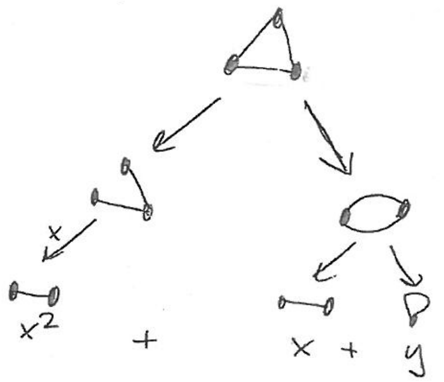
$$* \quad T(G) = \begin{cases} x\,T(G/e) & e \text{ bridge} \\ y\,T(G\setminus e) & e \text{ loop} \\ T(G\setminus e) + T(G/e) & e \text{ ordinary} \end{cases}$$

$* \quad T(\cdot) = 1$

$* \quad T(G \sqcup H) = T(G)\cdot T(H) = T(G \vee H)$

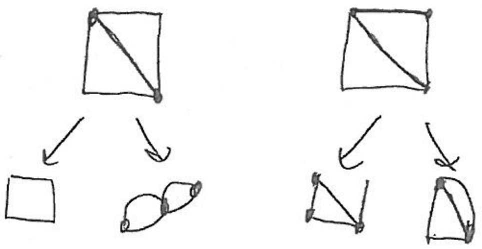
$T(\ldots) = T(\ldots)\cdot T(\ldots) = T(\ldots)$

Example



$T(\triangle) = x^2 + x + y$

- There is a choice of order of resolution of edges:



- <u>Thm:</u> (i) $T(G)$ is well-defined

(ii) $T(G) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)}(y-1)^{n(A)}$

- Tutte poly encodes lots of combinatorial info about a graph

$* \; (-1)^{|E|}\lambda^{k(G)}\,T(G; 1-\lambda, 0) =$ chromatic poly $= \#\ \lambda$-cols of graph

$* \; (-1)^{n(G)}\,T(G; 0, 1-\lambda) = \#$ nowhere zero λ-flows

$* \; T(G; x, \frac{1}{x}) \rightsquigarrow$ Jones / homfly poly

$* \; q^{k(G)} v^{r(G)}\,T(G; \frac{q}{v}+1, v+1) \rightsquigarrow q$ state potts. $v = e^{J\beta} - 1$

$* \; T(2,0) = \#$ acyclic orientations

$* \; T(1,0) = \#$ acyclic orientations w/ one specified source

$* \; T(0,2) = \#$ totally cyclic orientations (every edge in one cycle)

etc.

(lots of work: Handbook - 3 Z chapters)

Universality

- The Tutte poly is the most general deletion-contraction invariant for graphs, in the following sense.

Thm (Oxley, Welsh '79)

$\exists!$ map $U: \frac{\text{Graphs}}{\text{iso}} \to \mathbb{Z}[x,y,\sigma,\tau]$

s.t.

$$U((V,\emptyset)) = 1$$

$$U(G) = \begin{cases} x\, U(G\setminus e) & e \text{ bridge} \\ y\, U(G\setminus e) & e \text{ loop} \\ \sigma U(G\setminus e) + \tau U(G/e) & \text{otherwise} \end{cases}$$

Moreover

$$U(G) = \sigma^{n(G)} \tau^{r(G)} T\left(G; \frac{x}{\tau}, \frac{y}{\sigma}\right)$$

Ribbon graphs

- What is the "Tutte polynomial" of a ribbon graph?
- What do we mean by Tutte poly?
  * universal del-cont invariant
  * have deletion and contraction. Need to know cases. For this examine Universality for $T(G)$

- recognising bridges, loops and ordinary edges via deletion contraction.

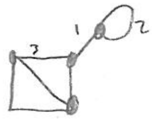
$$e^c := E \setminus e$$

$G \setminus e^c = \Big\{$  bridge $= b$, loop $= l$

$G/e^c = \Big\{$ 

- Say for $i,j \in \{l, b\}$

e is (i,j)-edge $\iff \begin{cases} G/e^c \text{ is } i \\ G\setminus e^c \text{ is } j \end{cases}$

- example: 

1 is (b,b)
2 is (l,l)
3 is (b,b)

- $(b,b) \iff$ bridge
  $(l,l) \iff$ loop
  (l,b) otherwise
  (b,l) impossible

- variables.
  a_i when G/e^c type i
  b_j when $G\setminus e^c$ type j

- Universality becomes.

$\exists!$ U st

$$U(V,\emptyset) = 1$$

$$U(G) = a_i U(G\setminus e) + b_j U(G/e) \quad \text{type}(i,j)$$

Moreover

$$U(G) = a_l^{|E|-r(G)} b_b^{r(G)} T\left(G; \frac{a_b}{b_b} + 1, \frac{b_l}{a_l} + 1\right)$$

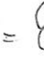
- Applying this to ribbon graphs

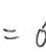
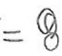
* $G\backslash e^c,\ G/e^c \in \{$  $\}$ (b, o, n)

* $i,j \in \{b, o, n\}$

e is $(i,j) \Leftrightarrow \begin{cases} G/e^c & \text{is } i \\ G\backslash e^c & \text{is } j \end{cases}$

<u>example</u>

$G/1^c =$ , $G\backslash 1^c =$  (n,o)

$G/2^c =$ , $G\backslash 2^c =$  (b,o)

- variables: a_i when G/e^c type i
  b_j when $G\backslash e^c$ type j

<u>Theorem:</u>

$\exists!$ invariant α satisfying

$\alpha(\text{edgeless}) = 1$

$$\alpha(G) = a_i\, \alpha(G\backslash e) + b_j\, \alpha(G/e) \quad \text{if } e \text{ is } (i,j)$$

if and only if

$$a_n = \sqrt{a_b a_o}, \quad b_n = \sqrt{b_b b_o}$$

in which case

$$\alpha = a_o^{|E|-\ell(G)}\, b_b^{\ell(G)} \sum_{A\subseteq E} \left(\frac{a_b}{b_o}\right)^{\ell(G)-\ell(A)} \left(\frac{b_b}{a_b}\right)^{|A|-\ell(A)}$$

- where

$$\ell(A) := \tfrac{1}{2}\left(|A| + v(A) - f(A)\right) = r(A) + \tfrac{1}{2}\gamma(A)$$

- So Define "Tutte poly" of a ribbon graph by

$$\widetilde{R}(G; x, y) := \sum_{A\subseteq E(G)} (x-1)^{\ell(G)-\ell(A)} (y-1)^{|A|-\ell(A)}$$

- Found in earlier lit as follows

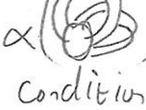
Bollobás-Riordan ('01, '02)

$$R(G; x, y, z) = \sum (x-1)^{r(G)-r(A)} y^{|A|-r(A)} z^{\gamma(A)}$$

$$\widetilde{R}(G; x+1, y+1) = x^{\frac{1}{2}\gamma(G)} R(G; x+1, y, 1/\sqrt{xy})$$

[• Remarks on this poly in lit.]

<u>proof:</u>

- Computing $\alpha($  $)$ in two ways gives the condition on the variables.
- Show state sum satisfies del-cont rels. by splitting sum $\sum_{\substack{A\subseteq E\\ e\notin A}}$ $\sum_{\substack{A\subseteq E\\ e\in A}}$

rewriting in terms of $\alpha(G\backslash e)$ $\alpha(G/e)$

$\square$