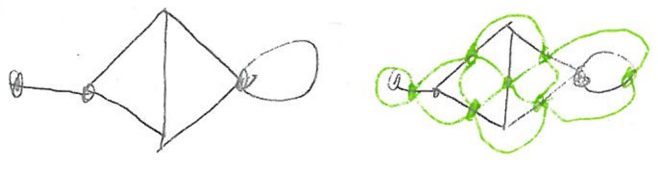
MEDIAL GRAPHS



G $\qquad\qquad$ G_m

- G graph in surface
  Its medial graph G_m is found
  * by placing one vertex on each edge
  * add edges by following face boundaries

- observe dual graphs have same medial graph

- Stronger relation holds:

Thm

$$G_m = H_m \iff H \in \{G, G^*\}$$

- let's pause and think what this means:

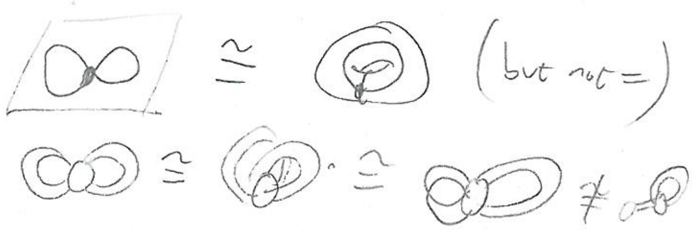
Concept of graph equivalence $\longleftrightarrow$ Concept of duality

- Can generate new dualities by looking at different concepts of graph equivalence.

- As an example let's determine duality generated by abstract graph iso

$G \cong H \iff$ embeddings of same abstract graph

eg



- proceed by understanding
  $G_m = H_m \iff H \in \{G, G^*\}$
  then dropping artificial restrictions.

Tait graphs

$F \subset \Sigma$ 4-regular.
If F can be face 2-coloured

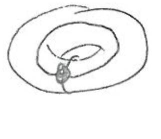
- faces bl or wh.
- adjacent faces different colours.

Tait graph: choose a bl col vertex in each black face, edges through vertices of F.

Let $\mathbb{T}(F) = \{\text{tait graphs of } F\}$

$|\mathbb{T}(F)| = 0, 1, 2$ ← two colourings
↖ the two may be iso.
↑ no Tait graph

eg

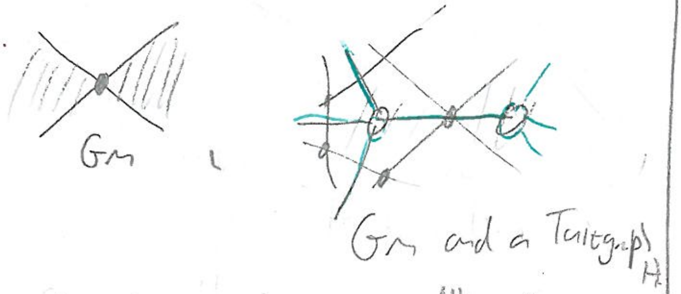


Lemma

$$G_m = H_m \iff H \in T(G_m)$$

pf:

- $H \in T(G_m)$



Clearly medial graph of H is G.

- H st $G_m = H_m$

  G_m a medial graph
  $\Rightarrow$ cb colourable (black faces correspond to vertices of G)

  $\Rightarrow$ Tait graphs exist and one of them is G.
  Since $H_m = G_m$ they must have the same tait graphs. These include G and H. □

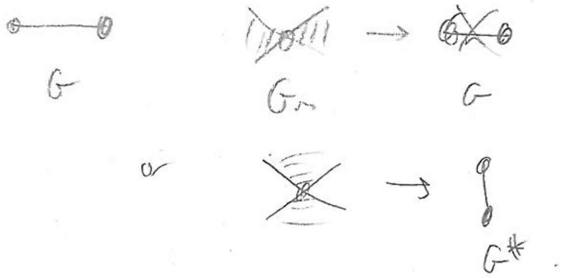
Lemma

$$H \in T(G_m) \iff H = G \text{ or } H = G^*$$

pf:

$\Leftarrow$ Tait graphs of G are G or G^*
G^* are G^* or G

$\Rightarrow$



Proof of Thm

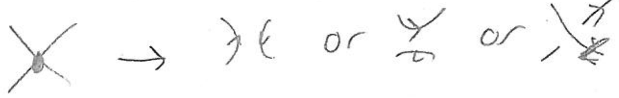
$$G_m = H_m \iff H \in T(G_m) \iff H \in \{G, G^*\}$$

Relaxing conditions

- Tait graphs in terms of arrow presentations

eg 

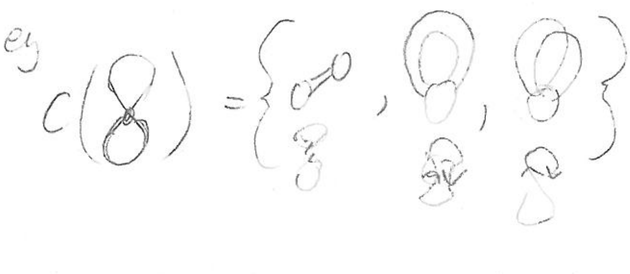
- Cycle family graphs
  - drop restriction



eg

- Denote by

  $C(F)$ set of cycle family graphs of F

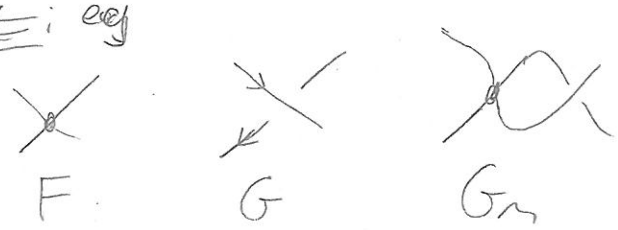
eg 

Thm F 4-reg

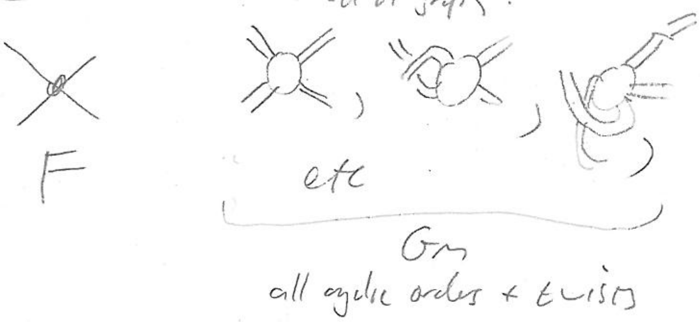
$G_m \cong F \iff G \in C(F)$

pf:

$\Leftarrow$: eg



$\Rightarrow$: Look what can arise as medial graphs



Construct G from these
check cycle family graphs

Def:

$G^{\tau(e)}$ partial petrial 

$G^{d(e)}$ partial dual

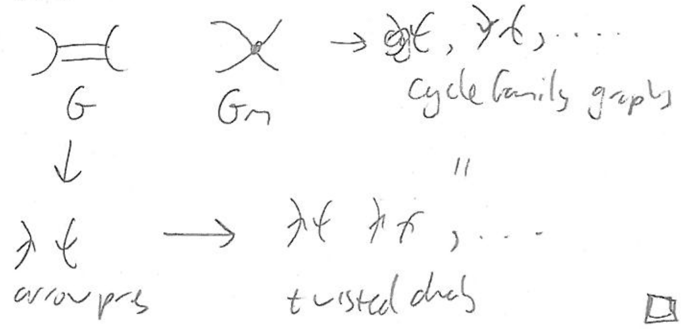
H is a twisted dual of G if it can be obtained from G by a sequence of partial petrials and partial duals.

eg 

Thm

$$C(G_m) = \left\{ \begin{array}{c} \text{twisted duals} \\ \text{of } G \end{array} \right\}$$

pf:



$\square$

combining

Thm

$G_m \cong H_m \iff H$ is a twisted dual of G.

We have seen

- G^* generated by equality of embedded graphs
- Twisted duals " " isomorphism of abstract graphs

Similar reasoning gives

- Partial duality " " equivalence of unsigned rotation systems

flips

same graph

Same or opposite cyclic ordering.