An introduction to ribbon graphs

Iain Moffatt
Royal Holloway, University of London

ALEA-Network Workshop
Bordeaux, 16-18 November 2015
Contents

What is a ribbon graph?
Subgraphs and minors
Polynomials
Duality
Medial graphs
Polynomials II
Knot theory
Matroids
Plan

What is a ribbon graph?

Subgraphs and minors

Polynomials

Duality

Medial graphs

Polynomials II

Knot theory

Matroids
What is a ribbon graph?

**Ribbon graph**

A “topological graph” with
- discs for vertices,
- ribbons for edges.

**Graph parameters**

\[ v(G) = \#(vertices), \quad e(G) = \#(edges), \quad k(G) = \#(cpts.) \]

**Topological parameters**

\[ f(G) = \#(boundary components), \quad g(G) = \text{genus} \]

\[ \gamma(G) = \text{Euler genus} = \begin{cases} 
2g(G) & \text{if orientable} \\
g(G) & \text{if non-orientable}
\end{cases} \]

**Euler’s Formula**

\[ v(G) - e(G) + f(G) = 2k(G) - \gamma(G) \]
What is a ribbon graph?

- **Cellularly embedded graph** – drawn in surface, faces are discs.
- Ribbon graphs describe cellularly embedded graphs.
- Considered up to homeomorphisms that preserve vertex-edge-structure and cyclic order at vertices.
- Warning:
  - Not embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.
  - No concept of a non-loop edge being “twisted”.
What is a ribbon graph?

Subgraphs and minors
Polynomials
Duality
Medial graphs
Polynomials II
Knot theory
Matroids

Arrow presentations

Edges as arrows

Arrow presentation

- Set of circles
- pairs of arrows on them
Plan

What is a ribbon graph?

Subgraphs and minors

Polynomials

Duality

Medial graphs

Polynomials II

Knot theory

Matroids
Deletion and subgraphs

Edge and vertex deletion

Ribbon subgraphs

- **ribbon subgraph** - delete edges and vertices.
- **spanning ribbon subgraph** - delete edges.

- Ribbon graphs are naturally closed under deletion.
- But big changes to corresponding cellurally embedded graph (which are not naturally closed under deletion).
Contraction

- Care needs to be taken:
  - Obvious if $e = (u, v)$ non-loop: make $e \cup u \cup v$ vertex.
  - Doesn’t work if $e$ is a loop.
- Three routes to a definition of contraction:
  - Arrow presentations.
  - Ribbon graphs.
  - Cellurally embedded graphs.

**Edge contraction**

To contract $e = (u, v)$:

- attach disc to each boundary-cpt. of $v \cup e \cup u$
- remove $v \cup e \cup u$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$G/e$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

- Caution: inconsistent with graph contraction!
Ribbon graph minors

Ribbon graph minor
Delete vertices, delete edges, contract edges.

An anti-chain of graphs but not of ribbon graphs:

\[ B = \left\{ \begin{array}{c}
\text{graphs}
\end{array} \right\} = \{B_3, B_5, B_7, \cdots \} \]

Conjecture (R.-S. theory for ribbon graphs?)
Every minor-closed family of ribbon graphs is characterised by a finite set of excluded minors.

- Why not a special case of Robertson-Seymour Theory?
Excluded minor characterisations

Proposition

\[ G \text{ is orientable } \iff \text{ no } -\text{minor} \]

\[ S_g := \{ G \mid \text{genus } g + 1, \text{ orient } G = \bigcup[1 \text{ vert.}, 1 \partial\text{-cpt}] \} \]

Theorem

Orientable \( G \) is of genus \( \leq g \) \iff \no \text{ minor in } S^o_g
Excluded minor characterisations

\[ S_{2k+1} := \{ G \mid \gamma(G) = 2k + 2, G = \bigcup [1 \text{ vert.}, 1 \partial\text{-cpt }] \} \]

\[ S_{2k} := \{ G \mid \gamma(G) = 2k + 1, G = \bigcup [1 \text{ vert.}, 1 \partial\text{-cpt }] \}
\text{ or } \gamma(G) = 2k + 2, \text{ orient } G = \bigcup [1 \text{ vert.}, 1 \partial\text{-cpt }] \} \]

Theorem

\[ G \text{ is of Euler genus } \leq g \iff \text{ no minor in } S_g \]
Plan

What is a ribbon graph?
Subgraphs and minors

Polynomials

Duality
Medial graphs
Polynomials II
Knot theory
Matroids
The Tutte polynomial, $T(G; x, y) \in \mathbb{Z}[x, y]$

$$T(G) = \begin{cases} 
1 & \text{if } G \text{ edgeless} \\
xT(G/e) & \text{if } e \text{ a bridge} \\
yT(G\setminus e) & \text{if } e \text{ a loop} \\
T(G\setminus e) + T(G/e) & \text{otherwise} 
\end{cases}$$

E.g., $T\left(\begin{array}{c} \cdot \\
\cdot \\
\bullet \end{array}\right) = x^2 + x + y$

Theorem

- $T(G)$ is well-defined.
- $T(G) = \sum_{A \subseteq E} (x - 1)^{r(G) - r(A)}(y - 1)^{|A| - r(A)}$

$T(G)$ encodes lots of information about a graph and appears in many places. (e.g., chromatic polynomial, flow polynomial, Ising model, Potts model, Jones polynomial, homfly-\text{pt polynomial},....)
A “Tutte polynomial” for ribbon graphs

- What do we mean by a Tutte polynomial?
- Universal deletion-contraction invariant.

Universality

\[ U(G) := \begin{cases} 
1 & \text{if } G \text{ edgeless} \\
xU(G/e) & \text{if } e \text{ a bridge} \\
yU(G\setminus e) & \text{if } e \text{ a loop} \\
\sigma U(G\setminus e) + \tau U(G/e) & \text{otherwise} 
\end{cases} \]

Then

\[ U(G) = \sigma^{|E| - r(G)} T(G; \frac{x}{\tau}, \frac{y}{\sigma}) \]

- \( i, j \in \{\text{bridge, loop}\} \)
- \( e \) is an \((i,j)\)-edge \(\iff\)
  \[ \begin{cases} 
  G/e^c \text{ is } i \\
  G\setminus e^c \text{ is } j 
\end{cases} \]
- \( U(G) = a_i U(G\setminus e) + b_j U(G/e) \) if \( e \) is \((i,j)\)-edge.
- \( U(G) = a_i^{|E| - r(G)} b_{\frac{a_b}{b_b}} \) if \( e \) is \((i,j)\)-edge.

- \( U(G) = a_i^{r(G)} b_{\frac{b_l}{a_l}} + 1, \frac{a_b}{b_b} + 1 \)
A “Tutte polynomial” for ribbon graphs

Apply to ribbon graphs:

- \( i,j \in \{ b,o,n \} \)
- \( e \) is an \((i,j)\)-edge \( \iff \begin{cases} G/e^c \text{ is } i \\ G\setminus e^c \text{ is } j \end{cases} \)
- \( \alpha(G) = a_i \alpha(G\setminus e) + b_j \alpha(G/e) \) if \( e \) is \((i,j)\)-edge.
- \( \alpha \) well-defined \( \iff a_n = (\sqrt{a_b a_o}) \) and \( b_n = (\sqrt{b_b b_o}) \)

“Tutte polynomial” of a ribbon graph

\[
\tilde{R}(G, x, y) = \sum_{A \subseteq E} (x - 1)^{\rho(G) - \rho(A)} (y - 1)^{|A| - \rho(A)},
\]

\( \rho(A) = r(A) + \gamma(A)/2 \)

- \( \alpha(G) = a_l^{\lvert E \rvert - \rho(G)} b_b^{\rho(G)} \tilde{R}(G; a_b b_b + 1, b_l a_l + 1) \)
- 2-variable specialisation of Bollobás-Riordan polynomial.
Plan

What is a ribbon graph?
Subgraphs and minors
Polynomials

Duality
Medial graphs
Polynomials II
Knot theory
Matroids
The geometric dual

The dual $G^*$ of an embedded graph $G$

- One vertex of $G^*$ in each face of $G$.
- One edge of $G^*$ when an edge separates faces in $G$.

$G = \text{Graph 1} \quad \Rightarrow \quad G^* = \text{Graph 2}$
The geometric dual

The dual $G^*$ of a ribbon graph $G$

- Fill in punctures of surface $G$ with vertices of $G^*$,
- then delete vertices of $G$ to get $G^*$.

$G = \begin{array}{c}
\text{embed} \\
\text{dual} \\
\text{expand} \\
\text{expand}
\end{array}$

$= G^*$
The geometric dual

The dual $G^*$ of an arrow presentation $G$.
Duality is a local operation! We can define:

**Partial duals**

$G^A$, the **partial dual** of $G$ with respect to $A \subseteq E(G)$:

dual only edges in $A$:

or
Partial duals of plane graphs

- When does $G$ have a partial dual in a given class?
- When is $G$ a partial dual of a plane graph?

**join (or connected sum)**

$G = P \lor Q$ if

- $G = P \sqcup Q$
- $P \cap Q = \{v\}$
- $E(P), E(Q)$ meet $v$ on disjoint arcs

**1-sum**

$G = P \oplus Q$ if

- $G = P \sqcup Q$
- $P \cap Q = \{v\}$
Partial duals of plane graphs

\[ G = P \oplus_3 Q \]

\[ P \subset \Sigma_P \text{ and } Q \subset \Sigma_Q \]

\[ P^* \subset \Sigma_P \text{ and } Q \subset \Sigma_Q. \]

\[ (P \oplus n Q)^E(P) \subset \Sigma \]
A ⊆ E(G) defines a **plane-biseparation** if the components of G|_A and G|_{A^c}

- are plane,
- intersect in at most one vertex.

**Theorem**

\[ G^A \text{ plane } \iff A \text{ defines a plane-biseparation.} \]
What is a ribbon graph?
Subgraphs and minors
Polynomials
23 Duality
Medial graphs
Polynomials II
Knot theory
Matroids

Partial duals of plane graphs

**plane-biseparation**

A \( \subseteq E(G) \) defines a **plane-biseparation** if the components of \( G|_A \) and \( G|_{A^c} \)

- are plane,
- intersect in at most one vertex.

Theorem

\( G^A \) plane \( \iff \) A defines a plane-biseparation.
Partial duals of plane graphs

**plane-biseparation**

$A \subseteq E(G)$ defines a plane-biseparation if the components of $G|_A$ and $G|_{A^c}$
- are plane,
- intersect in at most one vertex.

\[ A \subseteq E(G) \text{ defines a plane-biseparation if the components of } G|_A \text{ and } G|_{A^c} \]

\[ \text{are plane,} \]

\[ \text{intersect in at most one vertex.} \]

\[ P \quad Q \]

**Theorem**

$G^A$ plane $\iff$ $A$ defines a plane-biseparation.
Partial duals of plane graphs

plane-biseparation

$A \subseteq E(G)$ defines a **plane-biseparation** if the components of $G|_A$ and $G|_{A^c}$

- are plane,
- intersect in at most one vertex.

Theorem

$G^A \text{ plane } \iff A \text{ defines a plane-biseparation.}$
What is a ribbon graph?

Subgraphs and minors

Polynomials

Polynomials II

Knot theory

Matroids

Partial duals of plane graphs

**plane-biseparation**

$A \subseteq E(G)$ defines a **plane-biseparation** if the components of $G|_A$ and $G|_{A^c}$

- are plane,
- intersect in at most one vertex.

$P \subseteq Q$ defines a **plane-biseparation** if the components of $P|_A$ and $P|_{A^c}$

- are plane,
- intersect in at most one vertex.

**Theorem**

$G^A \text{ plane } \iff A \text{ defines a plane-biseparation.}$
**Partial duals of plane graphs**

**plane-biseparation**

\( A \subseteq E(G) \) defines a **plane-biseparation** if the components of \( G|_A \) and \( G|_{A^c} \)

- are plane,
- intersect in at most one vertex.

**Theorem**

\( G^A \) plane \( \iff \) \( A \) defines a plane-biseparation.
Partial duals of plane graphs

**plane-biseparation**

\[ A \subseteq E(G) \] defines a **plane-biseparation** if the components of \( G|_A \) and \( G|_{A^c} \)

- are plane,
- intersect in at most one vertex.

---

**Theorem**

\[ G^A \text{ plane } \iff A \text{ defines a plane-biseparation.} \]
Partial duals of plane graphs

**plane-biseparation**

$A \subseteq E(G)$ defines a **plane-biseparation** if the components of $G|_A$ and $G|_{A^c}$

- are plane,
- intersect in at most one vertex.

**Theorem**

$G^A$ plane $\iff A$ defines a plane-biseparation.
What is a ribbon graph?

Subgraphs and minors

Polynomials

Medial graphs

Polynomials II

Knot theory

Matroids

Partial duals of plane graphs

A subset $A \subseteq E(G)$ defines a **plane-biseparation** if the components of $G|_A$ and $G|_{A^c}$ are plane, and intersect in at most one vertex.

$P$ and $Q$ are examples of plane-biseparations.

**Theorem**

$G^A$ is plane $\iff A$ defines a plane-biseparation.
Partial duals of plane graphs

**Theorem**

*Partial dual of plane graph* $\iff$ *no minor isomorphic to*
Partial duals of plane graphs

If $G$ and $G^A$ are both plane, how are they related?

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>G plane.</strong> $G = H_1 \lor H_2 \lor \cdots \lor H_p.$</td>
</tr>
</tbody>
</table>

$$G^A \text{ plane } \iff A = \bigcup_{i \in I} H_i$$

**Dualling a summand:**

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>G plane.</strong></td>
</tr>
</tbody>
</table>

$$G^A \text{ plane } \iff \text{ obtained by dualling summands of } G$$
Plan

What is a ribbon graph?

Subgraphs and minors

Polynomials

Duality

Medial graphs

Polynomials II

Knot theory

Matroids
Medial graphs

The medial graph $G_m$ of a $G$

- One degree 4 vertex on each edge of $G \subset \Sigma$.
- Add edges by following face boundaries.

$G = \begin{tikzpicture}[baseline=-0.5ex]
  \begin{scope}[every node/.style={circle,draw,inner sep=1pt}]
    \node (A) at (0,0) {};
    \node (B) at (1,0) {};
    \node (C) at (0.5,0.866) {};
  \end{scope}
  \draw (A) -- (B) -- (C) -- (A);
\end{tikzpicture} \quad \rightarrow \quad \begin{tikzpicture}[baseline=-0.5ex]
  \begin{scope}[every node/.style={circle,draw,inner sep=1pt}]
    \node (A) at (0,0) {};
    \node (B) at (1,0) {};
    \node (C) at (0.5,0.866) {};
    \node (D) at (0.5,0.866) {};
    \node (E) at (0.25,0.633) {};
    \node (F) at (0.75,0.633) {};
    \node (G) at (0.5,1.732) {};
  \end{scope}
  \draw (A) -- (B) -- (C) -- (A) -- (D) -- (E) -- (F) -- (G);
\end{tikzpicture} \quad \rightarrow \quad \begin{tikzpicture}[baseline=-0.5ex]
  \begin{scope}[every node/.style={circle,draw,inner sep=1pt}]
    \node (A) at (0,0) {};
    \node (B) at (1,0) {};
    \node (C) at (0.5,0.866) {};
    \node (D) at (0.5,0.866) {};
    \node (E) at (0.25,0.633) {};
    \node (F) at (0.75,0.633) {};
    \node (G) at (0.5,1.732) {};
  \end{scope}
  \draw (A) -- (B) -- (C) -- (A) -- (D) -- (E) -- (F) -- (G);
\end{tikzpicture} \quad = \quad G_m$

Theorem

$G_m = H_m \iff H \in \{G, G^*\}$.

Concept of graph equivalence $\leftrightarrow$ Concept of graph duality

- Generate new graph dualities by changing concept of graph equivalence.
- Let’s determine duality generated by graph isomorphism.
- Proceed by understanding classical case.
Tait graphs, $\mathbb{T}(F)$

**Tait graphs**
- face 2-colour $F$,
- pick a colour & put vertices in faces of that colour.
- edges where faces touch

$F = \begin{array}{c} \text{triangle} \quad \rightarrow \quad \text{triangle} \quad \rightarrow \quad \text{triangle} \quad \rightarrow \quad \text{triangle} = F_{bl} \\
\end{array}$

$F = \begin{array}{c} \text{triangle} \quad \rightarrow \quad \text{triangle} \quad \rightarrow \quad \text{triangle} \quad \rightarrow \quad \text{loop} = F_{wh} \\
\end{array}$

**Tait graphs and medial graphs**
- $G_m = H_m \iff H \in \mathbb{T}(G_m)$.
- $H \in \mathbb{T}(G_m) \iff H = G$ or $H = G^*$.
- Together give $G_m = H_m \iff H \in \{G, G^*\}$. 
Consider Tait graphs as arrow presentations. Then drop the restrictions:

> Replace each \( v \) with one of:

\[
\mathcal{C} \left( \begin{array}{c}
\text{Cycle family graphs, } \mathcal{C}(F) \\
\end{array} \right) = \left\{ \begin{array}{c}
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array} \right\}
\]

**Theorem**

\[ G_m \cong F \iff G \in \mathcal{C}(F) \]
Twisted duals

Partial Petrial, $G^{τ(A)}$

“half-twist” edges is $A$:

Twisted dual

**Twisted dual:** a result of a sequence of partial duals and partial Petrials.
Twisted duals

**Theorem**

\[ C(G_m) = \{\text{twisted duals of } G\} \]

**Theorem**

\[ G_m = H_m \iff H \text{ a twisted dual of } G \]

- Concept of graph equivalence $\iff$ Concept of graph duality
  - equal as embedded graphs $\iff$ Geometric duality
  - equal as abstract graphs $\iff$ Twisted duality
  - equal as rotation systems $\iff$ partial duality
Plan

What is a ribbon graph?
Subgraphs and minors
Polynomials
Duality
Medial graphs
Polynomials II
Knot theory
Matroids
If only…

OK, I don’t really have time for this :-(
But here I would talk about:

▶ The Penrose polynomial.
▶ Various combinatorial interpretations of graph polynomials.
▶ How they are related via the transition polynomial and twisted duality.
The basics of knot theory

- **knot**: is a circle in $\mathbb{R}^3$.
- **link**: is disjoint circles in $\mathbb{R}^3$.

Considered up to **isotopy**: “you can push them around in space”.
The basics of knot theory

- **link diagram**: “nice drawing of link on plane”
- **Reidemeister’s Theorem**: $L = L' \iff$ diagrams related Reidemeister moves
The Jones polynomial

**Knot invariant:** $f$ s.t. $f(L) \neq f(L') \implies L \neq L'$.

The Kauffman bracket

- $\langle \bigotimes \rangle = A \langle \bigcirc \rangle + A^{-1} \langle \otimes \rangle$
- $\langle O \cup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$
- $\langle O \rangle = 1$

$$\langle \otimes \otimes \rangle = A \langle \bigcirc \bigcirc \rangle + A^{-1} \langle \otimes \otimes \rangle = A^2 \langle \bigotimes \rangle + \langle \bigcirc \bigcirc \rangle$$
$$+ \langle \bigotimes \bigotimes \rangle + A^2 \langle \bigotimes \bigotimes \rangle = (A^2 + A^{-2}) \langle O \rangle + 2 \langle O O \rangle = -A^2 - A^{-2}$$

The Jones polynomial

$$J(L) = ((-A)^{-3\omega(L)} \langle L \rangle)_{t^{1/2} = A^{-2}}$$

$$J\left( \bigotimes \bigotimes \right) = -t - t^2$$
The ribbon graphs of a link diagram

The ribbon graphs of a link diagram

-or-

All-A ribbon graph

A(G):
The Jones polynomial as a graph polynomial

Recall

- Tutte polynomial of a graph:
  \[ T(G) = \sum_{A \subseteq E} (x - 1)^{r(G) - r(A)} (y - 1)^{|A| - r(A)} \]

- Tutte polynomial of a ribbon graph:
  \[ \tilde{R}(G, x, y) = \sum_{A \subseteq E} (x - 1)^{\rho(G) - \rho(A)} (y - 1)^{|A| - \rho(A)} \]

Theorem

\[ J(D; t) = (-1)^w(D) t^{(3w(D) + |E| - 2\rho(A))/4} \]
\[ \quad \cdot (-t^{1/2} - t^{-1/2})^{k(\mathbb{A}) - \gamma(\mathbb{A})/2 - 1} \tilde{R}(G, -t, -t^{-1}) \]

Corollary

For D alternating

\[ J(D; t) = (-1)^w(D) t^{(3w(D) + |E| - 2\rho(A))/4} \]
\[ \quad \cdot (-t^{1/2} - t^{-1/2})^{k(\mathbb{A}) - 1} \tilde{T}(G, -t, -t^{-1}) \]
The ribbon graphs of links

- How are the ribbon graphs of a link diagram related?
- Which ribbon graphs describe links?
- How are diagrams with the same ribbon graphs related?

**Theorem**

\[ \text{Ribbon graphs } G \text{ and } G' \text{ represent the same link diagram } \iff \text{they are partial duals.} \]

**Theorem**

\[ \text{Ribbon graph represents a link diagram } \iff \text{it is partial dual of a plane graph.} \]
The ribbon graphs of links

- Excluded minors for partial duals of plane graphs gives:

**Theorem**

\[ G \text{ represents link diagram } \iff \text{ no minor isomorphic to } \]

\[ \text{, , } \]
The ribbon graphs of links

- Which links have the same ribbon graph?

- Dualling summands to move between plane partial duals gives:

**Theorem**

*D and D' represented by same set of ribbon graphs*

\[ D \sim D' \iff \text{related by} \]

- Isotopy class of a link is represented by a unique set of ribbon graphs.
Relating link digrams

▶ $G$ and $G^A$ both plane $\Longrightarrow$ related by

▶ Embed in $S^2$ and look at link diagram.
What is a ribbon graph?
Subgraphs and minors
Polynomials
Duality
Medial graphs
Polynomials II
Knot theory
Matroids
Matroids

- Vector spaces \(\leadsto\) matroids

### The Symmetric Exchange Axiom (SEA)

- \(\forall X, Y \in \mathcal{F},\) where \(\mathcal{F}\) a family of sets
- \(u \in X \triangle Y \implies \exists v \in X \triangle Y\) s.t. \(\{u, v\} \triangle X \in \mathcal{F}\)

### matroids (via bases)

- \(M = (E, \mathcal{B})\)
- \(\mathcal{B} \neq \emptyset,\) subsets of \(E\)
- \(\mathcal{B}\) satisfies SEA
- \(X, Y \in \mathcal{B} \implies |X| = |Y|\)

### Graphic matroid

- \(G = (V, E)\) a graph
- \(\mathcal{B} = \{\text{edge sets of spanning trees}\}\)
- **cycle matroid of** \(G: M(G) = (E, \mathcal{B})\)
Ribbon graphs and delta-matorids

- Cycle matroids don’t see topology.
- Want topological analogue of cycle matroid.

**trees**
- 1 boundary cpt.
- genus 0

\[ M(G) = (E, \{\{2\}, \{3\}\}) \]

**quasi-trees**
- 1 boundary cpt.

\[ D(G) = (E, \{\{1, 2, 3\}, \{2\}, \{3\}\}) \]

**\( D(G) \)**
- ribbon graph \( G = (V, E) \)
- \( \mathcal{F} = \{ \text{edge sets of spanning quasi-trees} \} \)
- \( D(G) = (E, \mathcal{F}) \)
Ribbon graphs and delta-matroids

**Matroids (via bases)**

\[ M = (E, B) \]

- \( B \neq \emptyset \), subsets of \( E \)
- \( B \) satisfies SEA
- \( X, Y \in B \implies |X| = |Y| \)

Cycle matroid (trees)

\[ M(G) = (E, \{\{2\}, \{3\}\}) \]

**Delta-matroids**

\[ D = (E, F) \]

- \( F \neq \emptyset \), subsets of \( E \)
- \( F \) satisfies SEA
- \( X, Y \in F \implies |X| = |Y| \)

\( \Delta \)-matroid (quasi-trees)

\[ D(G) = (E, \{\{1, 2, 3\}, \{2\}, \{3\}\}) \]

**Theorem**

\( D(G) \) is a delta-matroid.

Another example: non-singular principal submatrices.
Basic properties

max. min. matroids of $D$

- $\mathcal{F}_{\text{min}} = \{\text{feasible sets of min size}\}$
- $\mathcal{F}_{\text{max}} = \{\text{feasible sets of max size}\}$
- $D_{\text{min}} = (E, \mathcal{F}_{\text{min}})$ a matroid.
- $D_{\text{max}} = (E, \mathcal{F}_{\text{max}})$ a matroid.

- $D(G)_{\text{min}} = M(G)$
- $D(G)_{\text{max}} = M(G^*)^*$
- $\gamma(G) = r(D(G)_{\text{max}}) - r(D(G)_{\text{min}})$
- $G$ plane $\iff D(G) = M(G)$
- $G$ orientable $\iff$ all feasible sets have same parity
What is a ribbon graph?
Subgraphs and minors
Polynomials
Duality
Medial graphs
Polynomials II
Knot theory
Matroids

### Tutte polynomial

- \( M = (E, B) \) has rank function \( r \)
- \[ T(M; x, y) = \sum_{A \subseteq E} (x - 1)^{r(M) - r(A)} (y - 1)^{|A| - r(A)} \]
- \( T(G) = T(M(G)) \)

### Topological Tutte polynomial

- \( D_{\text{min}} \) rank function \( r_{\text{min}} \); \( D_{\text{max}} \) rank function \( r_{\text{max}} \)
- Average rank functions: \( \rho := \frac{1}{2}(r_{\text{max}} + r_{\text{min}}) \).
- \[ \tilde{T}(D; x, y) := \sum_{A \subseteq E} (x - 1)^{\rho(D) - \rho(A)} (y - 1)^{|A| - \rho(A)} \]
- \( \tilde{T}(G) = \tilde{T}(D(G)) \)

**Graph polynomial** ↔ **Matroid polynomial**

**Topological graph polynomial** ↔ **Delta-matroid polynomial**
Twists of matroids

Arguably the most fundamental operation in delta-matroid theory.

Twists

\[ D \ast A = (E, \{ F \triangle A \mid F \in \mathcal{F} \}) \]

Example:
\[ M = (\{1, 2, 3\}, \{\{1\}, \{2\}\}) \]
\[ M^* = M \ast E = (\{1, 2, 3\}, \{\{2, 3\}, \{1, 3\}\}) \]
\[ M \ast \{2, 3\} = (\{1, 2, 3\}, \{\{1, 2, 3\}, \{3\}\}) \]

Facts

\[ D^* = D \ast E \]
\[ G \text{ plane, } (M(G))^* = M(G^*) \]
\[ \text{All } G, \ (D(G))^* = D(G^*) \]
\[ D(G) \ast A = D(G^A) \]
Twists of matroids

- When is $D$ a twist of a matroid?
- Translate into ribbon graphs.

**Ribbon graph version**

Partial dual of plane graph $\iff$ no minor isomorphic to

**Theorem**

Partial dual of matroid $\iff$ no minor isomorphic to

$D\begin{pmatrix} \circ \end{pmatrix}, D\begin{pmatrix} \circ \circ \circ \end{pmatrix}, D\begin{pmatrix} \circ \circ \circ \circ \circ \end{pmatrix}$

- Ribbon graphs guide delta-matroid theory.