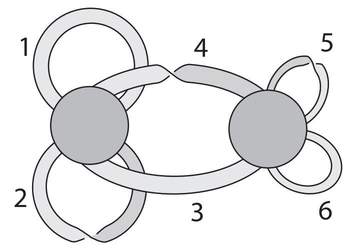
1. Consider the ribbon graph G given by 

.

   Compute the following:

   (a) $v(G)$, $e(G)$, $f(G)$, $k(G)$, $\gamma(G)$.
   (b) G/e, for each of $e = 1, 2, \ldots, 6$
   (c) G^e, for each of $e = 1, 2, \ldots, 6$
   (d) G^A for $A = \{3, 4, 5\}$, $A = \{1, 2, 6\}$, and $A = E(G)$.

2. Let G be the cellularly embedded graph . Suppose that $E(G) = \{a, b, c\}$ with a being the loop. What are the cellularly embedded graphs corresponding to $G \backslash b$, G/a, G/b, $G/\{b, c\}$?

3. Verify that $G/e = G^e \backslash e$.

4. Show that for any ribbon graph G, $\gamma(G) - \gamma(G \backslash e) \in \{0, 1, 2\}$, but if G is orientable $\gamma(G) - \gamma(G \backslash e) \in \{0, 2\}$.

5. We saw the "ribbon graph version" of Euler's Formula:

$$v(G) - e(G) + f(G) = 2k(G) - \gamma(A).$$

   If you haven't worked with ribbon graphs before, then you probably haven't seen this version of the formula. However, you probably have seen the version for surfaces, which is as follows. Given a closed connected surface Σ. Triangulate Σ. Let v_t, e_t, f_t be the numbers of vertices, edges and faces in this triangulation. Then

$$v_t - e_t + f_t = 2 - \gamma(\Sigma).$$

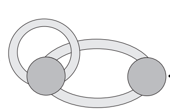
   ($v_t - e_t + f_t$ is the Euler Characteristic, $\chi(\Sigma)$, of Σ.) For a surface with boundary, this becomes

$$v_t - e_t + f_t = 2 - \gamma(\Sigma) - b$$

   where b is the number of boundary components. (Since each boundary component corresponds to an "empty triangle".) Derive the ribbon graph version of Euler's Formula from the surface version.

6. Show that $f(G \backslash A) = f(G^* \backslash A^c)$.

7. Let δ be the partial dual operation, and τ be the partial Petrial operation. Verify that the symmetric group $\mathfrak{S}_3 = \langle \delta, \tau \,|\, \delta^2, \tau^2, (\delta\tau)^3 \rangle$ acts on the set ribbon graphs with a distinguished edge by $g(G, \{e\}) = (G^{g(e)}, \{e\})$. This show that twisted duality can be constructed through group actions.

8. Compute $\tilde{R}$ of .

9. The three cases for the deletion-contraction relation of the Tutte polynomial are given in terms of edge types (loop, bridge, ordinary). Express the deletion-contraction relation for $\tilde{R}$ in terms of edge types (trivial orientable loop, trivial non-orientable loop, non-trivial orientable loop, non-trivial non-orientable loop, bridge, and ordinary).

10. Prove that

    (a) $\tilde{R}(G \vee H) = \tilde{R}(G) \cdot \tilde{R}(H)$

    (b) $\tilde{R}(G; x, y) = \tilde{R}(G^*; y, x)$

    (c) Suppose H is obtained from G by replacing each edge as a pair of parallel edges. Express $\tilde{R}(H)$ in terms of $\tilde{R}(G)$. (More generally, the same question but replacing each edge with any fixed ribbon graph.)

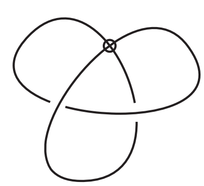
11. We saw a characterisation of partial duals of plane graphs in terms of plane-biseparations. Use the approach from the lectures to find an analogous characterisation partial duals of graph in the real projective plane (i.e., of Euler genus 1).

12. Let $F = (V, E)$ be a connected 4-regular graph. Each vertex v of F is incident with exactly four half-edges. A *transition* τ_v at a vertex v is a partition of the half-edges at v, and a *transition system*, $\tau := \{\tau_v \mid v \in V\}$ of F is a choice of transition at each of its vertices. Since F is 4-regular, at each vertex there are three transitions. Choose exactly two transitions τ_v and τ_v' at each vertex, and consider the set $\mathcal{T}(F)$ consisting of all transition systems of F in which the transition at each vertex v is one of the distinguished transitions, τ_v or τ_v'. An element of $\mathcal{T}$ is called a *transversal*.

    (a) Let G be a ribbon graph and G_m be its medial graph. Show that there is a natural way to construct $\mathcal{T}(G_m)$ and a transversal, so that the transition states $\mathcal{T}(G_m)$ correspond to the spanning ribbon subgraphs of G, with the transversal corresponding to E.

    (b) Show that for any connected 4-regular graph F, and any $\mathcal{T}(F)$ and transversal, there is some ribbon graph G such that $\mathcal{T}(G_m) = \mathcal{T}(F)$ and the transversal corresponds with G, as in the first part of the question.

13. A virtual link diagram is essentially a link diagram on the plane in which some of the crossings are marked as *virtual crossings*. For example



    is a virtual link diagram. The virtual crossings marked by circles. The definition of the Kauffan bracket extends to virtual links provided we allow virtual crossings in our "crossingless" links, and only resolve "classical" crossings.

    (a) Verify that for the above virtual link $\langle D \rangle = A^2 + 1 + A^{-4}$.

    (b) Compute the all-A ribbon graph of the above virtual link.

    (c) Express $\langle D \rangle$ in terms of $\tilde{R}$.