Permanents and the Jones polynomial

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Permanents

**Determinant of an** $n \times n$ matrix $A = [a_{ij}]$

\[
\text{det}(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i \sigma(i)}
\]

\[
\text{det}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei - afh + dch - dbi + gbf - gce
\]

**Permanent of an** $n \times n$ matrix $A = [a_{ij}]$

\[
\text{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \sigma(i)}
\]

\[
\text{per}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + afh + dch + dbi + gbf + gce
\]
Permanents

**Determinant of an** \( n \times n \) **matrix** \( A = [a_{ij}] \)

\[
\det(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i \sigma(i)}
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**Permanent of an** \( n \times n \) **matrix** \( A = [a_{ij}] \)

\[
\text{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \sigma(i)}
\]

\[
\text{per} \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = aei + afh + dch + dbi + gbf + gce
\]
Permanents: complexity

- **P** solvable in polynomial time. 
  e.g. Given a cycle of $G$, is it a Hamiltonian?
- **NP** solution can be *verified* in polynomial time. 
  e.g. does a graph have a Hamiltonian cycle?
- **#P** number of solutions to a problem in **NP**. 
  e.g. how many Hamiltonian cycles does a graph have?

Facts

- Computing det($A$) is **P**.
- Computing per($A$) is **#P**.

A fun fact

MacMahon Master Theorem gives per($A$) = $[x^1]$($\det(I - XA)$)$^{-1}$, 
where $X = \text{diag}(x_1 \cdots x_n)$, and $[x^1]$ is the coefficient of $x_1 \cdots x_n$. 
**Q**: Does it follow that **P** = **#P**?
**Definitions**

- **$G$** a directed graph (digraph), i.e. edges are directed.
- $A = [a_{ij}]$ is the adjacency matrix of $G$ if $a_{ij}$ is the number of edges from $i$ to $j$.
- A cycle cover of $G$ is a collection of vertex-disjoint directed cycles that covers all vertices.

**Theorem**

$\text{per}(A) = \#\text{cycle covers}.$
**Permanents: combinatorial interpretation**

- **$G$** a **weighted digraph**, i.e. every edge was a weight.
- **$A = weighted adjacency matrix**, i.e. $a_{ij} =$ weight of $i$ to $j$ edges.
- Every cycle cover has a weight.

$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & d + e \\ 0 & 0 & f & c \\ a & 0 & 0 & 0 \end{bmatrix}$

$\text{per}(A) = a^2 bc + a^2 df + a^2 ef$

**Theorem**

$\text{per}(A) = \text{generating function for cycle covers w.r.t. weights.}$

$= \sum_{\text{cyc. cov.} C} \prod_{e \in C} w(e)$
Permanents: combinatorial interpretation

- $G$ a weighted digraph, i.e. every edge was a weight.
- $A = \text{weighted adjacency matrix}$, i.e. $a_{ij} = \text{weight of } i \text{ to } j \text{ edges}$.
- Every cycle cover has a weight.

$$A = \begin{bmatrix}
0 & a & 0 & 0 \\
0 & 0 & b & d + e \\
0 & 0 & f & c \\
a & 0 & 0 & 0
\end{bmatrix}$$

$$\text{per}(A) = a^2 bc + a^2 df + a^2 ef$$

Theorem

$$\text{per}(A) = \text{generating function for cycle covers w.r.t. weights.}$$

$$= \sum_{\text{cyc.cov.}} \prod_{e \in C} w(e)$$
Knots and links

- **A knot** is a circle, $S^1$, sitting in 3-space $\mathbb{R}^3$.
- **A link** is a number of disjoint circles in 3-space $\mathbb{R}^3$.

Knots and links are considered up to ambient isotopy. You can “move them round in space, but you can’t cut or glue them”.

![Diagram of knots and links](image-url)
Knots and links

- The fundamental problem in knot theory is to determine whether or not two links are isotopic.

To do this we need a way to tell knots and links apart.

Definition

A **knot invariant** is a function \( F : \frac{\text{links}}{\text{isotopy}} \rightarrow S \) such that

\[
F(L) \neq F(L') \implies L \neq L'
\]
The Jones polynomial

The 

Jones polynomial

is knot invariant defined by the relations

- \( q^2 J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

dl
\end{tikzpicture}}
\end{array} \right) - q^{-2} J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right) = (q - q^{-1}) J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right), \)

- \( J(QQ\cdots Q) = (q + q^{-1})^k. \)

\[
J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right) = q^{-4} J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right) + q^{-2}(q - q^{-1}) J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right)
\]
\[
= q^{-4} J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right) + q^{-2}(q - q^{-1}) J \left( \begin{array}{c}
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\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right) + q^{-2}(q - q^{-1}) J \left( \begin{array}{c}
\text{\begin{tikzpicture}
  
  

\end{tikzpicture}}
\end{array} \right)
\]
\[
= [q^{-4}(q + q^{-1})^2] + [q^{-2}(q - q^{-1})(q + q^{-1})]
\]
\[
= q^{-6} + q^{-4} + q^{-2} + 1
\]
The Jones polynomial

The **Jones polynomial** is knot invariant defined by the relations

\[ q^2 J \left( \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right) - q^{-2} J \left( \begin{array}{c} \leftarrow \\ \rightarrow \end{array} \right) = (q - q^{-1}) J \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right), \]

\[ J(QQ \cdots Q) = (q + q^{-1})^k. \]

\[ J \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) = q^{-6} + q^{-4} + q^{-2} + 1 \]

\[ J \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) = q^{-4} J \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) + q^{-2}(q - q^{-1}) J \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) \]

\[ = q^{-4} J \left( \begin{array}{c} Q \\ Q \end{array} \right) + q^{-2}(q - q^{-1}) J \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right) \]

\[ = q^{-4}(q + q^{-1}) + q^{-2}(q - q^{-1})(q^{-6} + q^{-4} + q^{-2} + 1) \]

\[ = q^{-1} + 3q^{-3} + 3q^{-5} + 2q^{-7} + q^{-9} \]
The Jones polynomial

\[ J \left( \begin{array}{c}
\text{\includegraphics{link1}}
\end{array} \right) = q^{-1} + 3q^{-3} + 3q^{-5} + 2q^{-7} + q^{-9} \]

\[ J \left( \begin{array}{c}
\text{\includegraphics{link2}}
\end{array} \right) = q + 3q^3 + 3q^5 + 2q^7 + q^9 \]

⇒ \begin{array}{c}
\text{\includegraphics{link1}} \\
\text{\includegraphics{link2}}
\end{array} \neq \begin{array}{c}
\text{\includegraphics{link1}} \\
\text{\includegraphics{link2}}
\end{array}

- The Jones polynomial does not distinguish all links:

\[ J \left( \begin{array}{c}
\text{\includegraphics{link3}}
\end{array} \right) = J \left( \begin{array}{c}
\text{\includegraphics{link4}}
\end{array} \right) , \quad \text{but } \text{\includegraphics{link3}} \neq \text{\includegraphics{link4}} \]

- \( J(L) = J(\text{\includegraphics{link5}}) \quad \implies \quad L = \text{\includegraphics{link5}} \)

- Open question: \( J(L) = J(\text{\includegraphics{link6}}) \quad \implies \quad L = \text{\includegraphics{link6}} \)
Recap

Permanents & cycle covers: \( \text{per}(A) = \sum_{\text{cyc.cov.C}} \prod_{e \in C} w(e) \)

\[ A = \begin{bmatrix}
0 & a & 0 & 0 \\
0 & 0 & b & d + e \\
0 & 0 & f & c \\
a & 0 & 0 & 0
\end{bmatrix} \]

\[ \text{per}(A) = a^2 bc + a^2 df + a^2 ef \]

The Jones polynomial: a knot invariant

\[ J \left( \begin{array}{c}
\end{array} \right) \neq J \left( \begin{array}{c}
\end{array} \right) \Rightarrow \begin{array}{c}
\end{array} \neq \begin{array}{c}
\end{array} \]
Theorem (M. & Loebl)

The Jones polynomial is a permanent:

\[ J(L) = q^{\text{rot}(L) - 2\omega(L)} \text{ per}(M_L), \]

- \( \text{rot}(L) \) = rotation number of link.
- \( \omega(L) = (\# \leftarrow \rightarrow \text{ crossings}) - (\# \rightarrow \leftarrow \text{ crossings}) \)

Approach

- Construct a digraph from a link.
- Express \( J(L) \) in terms of cycle covers of a digraph.
- Use fact that permanents enumerate cycle covers.
- **Expositional simplification:** for this talk, all crossings \( \leftarrow \rightarrow \).
Step 1: connection with statistical mechanics

- **Ice-type models** are used in statistical physics to study the energy levels of crystal lattices with hydrogen bonds such as ice.

Theorem: Jones polynomial is an ice type model (Turaev, Jones)

\[
J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s)
\]

- \(\omega(L) = \left( \text{\# crossing with overpass over underpass} \right) - \left( \text{\# crossing with underpass overpass} \right)\)
- The sum is over \(\{0, 1\}\)-labelings of the arcs in a link
- \(\text{rot}_i\) travel round the \(i\)-labelled curves, count number of revolutions.
- \(R_v(s)\) look at each crossing:

<table>
<thead>
<tr>
<th>col.</th>
<th>(q)</th>
<th>(q - q^{-1})</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>(q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q)</td>
<td>(q)</td>
<td>(q - q^{-1})</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(q)</td>
</tr>
<tr>
<td>(q^{-1})</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(q - q^{-1})</td>
<td>(q^{-1})</td>
<td></td>
</tr>
</tbody>
</table>
An example

**Jones polynomial**

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]

sum over \{0,1\}-colourings of arcs between crossings

**Vertex weights** \( R_v(s) \)

<table>
<thead>
<tr>
<th>( \text{rot}_0(s) )</th>
<th>( \text{rot}_1(s) )</th>
<th>( q^{\text{rot}_0 - \text{rot}_1} )</th>
<th>( \prod R_v(s) )</th>
<th>( q^{-2\omega(L)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>0</td>
<td>( q^0 )</td>
<td>( q^2 )</td>
<td>( q^2 )</td>
</tr>
<tr>
<td>( q - q^{-1} )</td>
<td>-1</td>
<td>( q^{-2} )</td>
<td>( q^2 )</td>
<td>( q^0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1-1=0</td>
<td>1-1=0</td>
<td>1-1=0</td>
<td>1-1=0</td>
<td>1-1=0</td>
</tr>
</tbody>
</table>

\[ J(\text{[diagram]}) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} \]

\[ = q^{-6} + q^{-4} + q^{-2} + 1 \]
An example

Jones polynomial

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]

rotation number of 0-coloured components

vertex weights \( R_v(s) \)

| \( \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} \) | \( \begin{array}{c}
\begin{array}{c}
\downarrow \\
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\end{array} \) | \( \begin{array}{c}
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\end{array}
\end{array} \) | \( \begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
\end{array} \) | \( \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} \) |
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<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{rot}_0(s) )</td>
<td>1-1=0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \text{rot}_1(s) )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1-1=0</td>
</tr>
<tr>
<td>( q^{\text{rot}_0 - \text{rot}_1} )</td>
<td>( q^0 )</td>
<td>( q^2 )</td>
<td>( q^{-2} )</td>
<td>( q^2 )</td>
<td>( q^0 )</td>
<td>( q^0 )</td>
</tr>
<tr>
<td>( \prod_v R_v(s) )</td>
<td>( q^2 )</td>
<td>( (q - q^{-1})^2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( q^2 )</td>
</tr>
<tr>
<td>( q^{-2\omega(L)} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
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</tr>
</tbody>
</table>

\[ J(L) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} \]

\[ = q^{-6} + q^{-4} + q^{-2} + 1 \]
An example

### Jones polynomial

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s)} - q^{\text{rot}_1(s)} \prod_v R_v(s) \]

**rotation number of 1-coloured components**

### Vertex weights \( R_v(s) \)

<table>
<thead>
<tr>
<th>( R_v(s) )</th>
<th>( q )</th>
<th>( q - q^{-1} )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{rot}_0(s) )</td>
<td>1-1=0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \text{rot}_1(s) )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( q^{\text{rot}_0-\text{rot}_1} )</td>
<td>( q^0 )</td>
<td>( q^2 )</td>
<td>( q^{-2} )</td>
</tr>
<tr>
<td>( \prod R_v(s) )</td>
<td>( q^2 )</td>
<td>( (q - q^{-1})^2 )</td>
<td>1</td>
</tr>
<tr>
<td>( q^{-2\omega(L)} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
</tr>
</tbody>
</table>

\[ J(\text{circle}) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} = q^{-6} + q^{-4} + q^{-2} + 1 \]
An example

Jones polynomial

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]

vertex weights \( R_v(s) \)

<table>
<thead>
<tr>
<th>( R_v(s) )</th>
<th>1-1=0</th>
<th>1</th>
<th>-1</th>
<th>1</th>
<th>-1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{rot}_0(s) )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1-1=0</td>
</tr>
<tr>
<td>( \text{rot}_1(s) )</td>
<td>1-1=0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( q^{\text{rot}_0 - \text{rot}_1} )</td>
<td>( q^0 )</td>
<td>( q^2 )</td>
<td>( q^{-2} )</td>
<td>( q^2 )</td>
<td>( q^0 )</td>
<td>( q^0 )</td>
</tr>
<tr>
<td>( \prod R_v(s) )</td>
<td>( q^2 )</td>
<td>( (q - q^{-1})^2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>( q^{-2\omega(L)} )</td>
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<td>( q^{-4} )</td>
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<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
</tr>
</tbody>
</table>

\[ J\left( \begin{array}{c} \text{ \rotatebox{90}{\Large \vcheck}} \\ \end{array} \right) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} \]

\[ = q^{-6} + q^{-4} + q^{-2} + 1 \]
An example

### Jones polynomial

$$J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s)$$

**Product of vertex weights read from table**

<table>
<thead>
<tr>
<th>$\text{rot}_0(s)$</th>
<th>$\text{rot}_1(s)$</th>
<th>$q^{\text{rot}_0(s) - \text{rot}_1(s)}$</th>
<th>$\prod_v R_v(s)$</th>
<th>$q^{-2\omega(L)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1=0</td>
<td>0</td>
<td>$q^0$</td>
<td>$q^2$</td>
<td>$q^{-4}$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$q^2$</td>
<td>$(q - q^{-1})^2$</td>
<td>$q^{-4}$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$q^{-2}$</td>
<td>1</td>
<td>$q^{-4}$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$q^2$</td>
<td>0</td>
<td>$q^{-4}$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$q^0$</td>
<td>$q^2$</td>
<td>$q^{-4}$</td>
</tr>
<tr>
<td>0</td>
<td>1-1=0</td>
<td>$q^0$</td>
<td>0</td>
<td>$q^{-4}$</td>
</tr>
</tbody>
</table>

$$J\left( \right) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} = q^{-6} + q^{-4} + q^{-2} + 1$$
An example

**Jones polynomial**

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]

**Vertex weights** \( R_v(s) \)

<table>
<thead>
<tr>
<th>( \rotate[0])</th>
<th>( \rotate[1])</th>
<th>( )</th>
<th>( )</th>
<th>( )</th>
<th>( )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>( q - q^{-1} )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( q )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{array}{c|c|c|c|c|c}
\rotate[0](s) & 1-1=0 & 1 & -1 & 1 & -1 & 0 \\
\rotate[1](s) & 0 & -1 & 1 & -1 & 1 & 1-1=0 \\
q^{\text{rot}_0 - \text{rot}_1} & q^0 & q^2 & q^{-2} & q^2 & q^0 & q^0 \\
\prod R_v(s) & q^2 & (q - q^{-1})^2 & 1 & 1 & 0 & q^2 \\
q^{-2\omega(L)} & q^{-4} & q^{-4} & q^{-4} & q^{-4} & q^{-4} & q^{-4} \\
\end{array} \]

\[ J\left( \begin{array}{c}
\rotate[0]
\end{array} \right) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} \]

\[ = q^{-6} + q^{-4} + q^{-2} + 1 \]
An example

Jones polynomial

\[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]

Vertex weights \( R_v(s) \)

<table>
<thead>
<tr>
<th></th>
<th>( q )</th>
<th>( q - q^{-1} )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{rot}_0(s) )</td>
<td>1 - 1 = 0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \text{rot}_1(s) )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( q^{\text{rot}_0 - \text{rot}_1} )</td>
<td>( q^0 )</td>
<td>( q^2 )</td>
<td>( q^{-2} )</td>
</tr>
<tr>
<td>( \prod R_v(s) )</td>
<td>( q^2 )</td>
<td>( (q - q^{-1})^2 )</td>
<td>1</td>
</tr>
<tr>
<td>( q^{-2\omega(L)} )</td>
<td>( q^{-4} )</td>
<td>( q^{-4} )</td>
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</tr>
</tbody>
</table>

\[ J\left( \begin{array}{c} \circ \end{array} \right) = q^{-2} + q^{-2}(q - q^{-1})^2 + q^{-6} + q^{-2} + 0 + q^{-2} \]

\[ = q^{-6} + q^{-4} + q^{-2} + 1 \]
Step 2: making contact with cycle covers

In principle:

- Ignore rotation numbers for the moment.
- Express $\sum_s \prod_v R_v(s)$ in terms of weighted cycle covers.
- Replace each crossing of link with a digraph:
  
  ![Diagram of crossing replacement]
  
  - $\{\text{Cols. of link}\} \leftrightarrow \{\text{Cycle covers of digraph}\}$ (follow blue paths):

  ![Diagram of cyclic covers]

  - Weight edges s.t. (weight cycle cover) = (weight crossing type)
Step 2: making contact with cycle covers

In principle:
- Ignore rotation numbers for the moment.
- Express $\sum_s \prod_v R_v(s)$ in terms of weighted cycle covers.
- Replace each crossing of link with a digraph:

$$q \leftrightarrow 1 \leftrightarrow 1 \leftrightarrow q - q^{-1} \leftrightarrow 0 \leftrightarrow q$$

- Weight edges s.t. (weight cycle cover) = (weight crossing type)

In practice: Doesn’t quite work!
Step 2: making contact with cycle covers

A more complicated digraph is needed.

- Replace crossings:

- \{\text{weighted cols. of link}\} \leftrightarrow \{\text{weighted cycle covers of digraph}\}, e.g.

\[
0 = \frac{1}{2} \times q + -\frac{1}{2} q
\]
Step 2: making contact with cycle covers

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- Replace crossings:

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We have constructed a digraph from a link such that

\[\sum_{s} \prod_{v} R_v(s) = \sum_{\text{cyc. cov.}} \prod_{e} w(e)\]
Step 3: bringing in the rotation numbers

- \[ J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \]
- \( \text{rot}_0(s) - \text{rot}_1(s) = [\text{rot}_0(s) + \text{rot}_1(s)] - 2\text{rot}_1(s) = \text{rot}(L) - 2\text{rot}_1(s) \)
- Blue paths of \( L \) give \( \text{rot}_1(s) \)
- Blue paths ↔ edges between gadgets.
- Weight edges with \((-2) \times \text{rotation number}\)

\[ \text{rot}(L) = 0 \quad \text{digraph} \quad \text{rot}_0(s) = 1 \quad \text{rot}_1(s) = -1 \quad \sum_e \text{rot}(e) = -1 \]
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Replace crossings:

**Lemma**

\[
\sum_{\text{col. of } L} \prod_v q^{-2\text{rot}_1(L)} R_v(s) = \sum_{\text{cyc. cov.}} \prod_e w(e)
\]
Step 3: bringing in the rotation numbers

\[
q^{-2} q (q + q^{-1})/2 + q^{-2} q (q - q^{-1})/2 = q^{-2}
\]
Step 4: putting it all together

What we know

1. \( J(L) = q^{-2\omega(L)} \sum_s q^{\text{rot}_0(s) - \text{rot}_1(s)} \prod_v R_v(s) \)
2. \( \sum_{\text{col. of } L} \prod_v q^{-2\text{rot}_1(L)} R_v(s) = \sum_{\text{cyc. cov.}} \prod_e w(e) \)
3. \( \sum_{\text{cyc. cov.}} \prod_e w(e) = \text{per}(A) \)

Theorem (M. & Loebl)

The Jones polynomial is a permanent:

\[ J(L) = q^{\text{rot}(L) - 2\omega(L)} \text{per}(M_L), \]

\( M_L \) is the adjacency matrix of the weighted digraph.
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An example

\[ L = \]

\[ M_L = \begin{bmatrix} 0 & 0 & q - q^{-1} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{2}(q + q^{-1}) & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ q^{-2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & q^2 & q & 0 & -\frac{1}{2}q & 0 & 0 \end{bmatrix} \]

\[ \text{per}(M_L) = q^3 + q \]

\[ J \begin{bmatrix} \text{\includegraphics{rot.png}} \end{bmatrix} = q^{\text{rot}(L) - 2\omega(L)} \text{per}(M_L) = q^{0-2}(q^3 + q) = q + q^{-1} \]
Conclusions; i.e., so what?

- We used
  - combinatorial interpretation of the permanent (enumerating cycle covers)
  - statistical mechanical construction of the Jones polynomial

  to express the Jones polynomial as a permanent:

\[ J(L) = q^{\text{rot}(L)} - 2^{\omega(L)} \text{per}(M_L). \]

- Applications to computation:
  - 😞Bad news: computing \( \text{per}(A) \) is \( \mathbb{NP} \).
  - 😊Good news: there are approximation algorithms for permanents.
    - (Note: approximating perm of +ve matrix is poly time, but not poly time for an arbitrary matrix.)

- Question: does this lead to an efficient Mont-Carlo algorithm for approximating the Jones polynomial? (Work in progress joint with Martin Loebl and Petr Plechac.)
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Thanks!

Reference for this work: