Characterizing the ribbon graphs of knots

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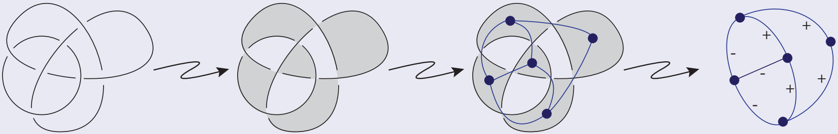
Graphs and link diagrams

There is a well known way to get a plane graph from a link diagram:

Tait graphs



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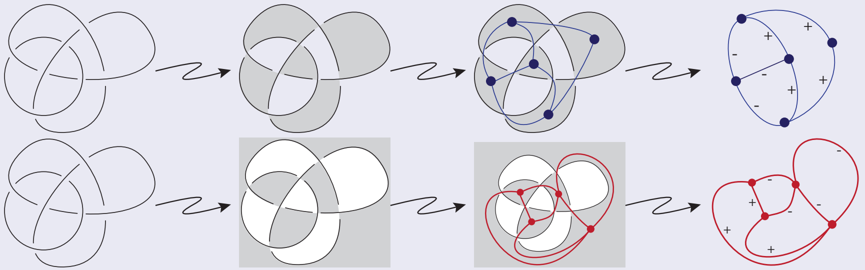

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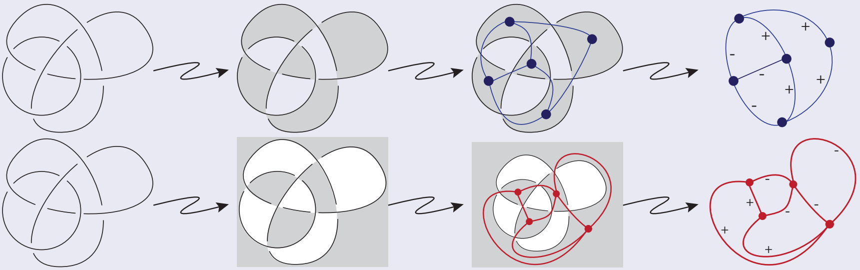
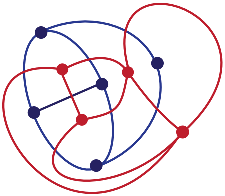


Properties

- The Tait graphs of a link diagram are duals.
- Every plane graph describes a link diagram.
- A Tait graph describes a unique link diagram.

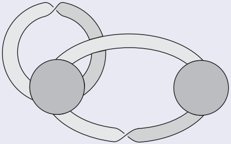


Ribbon graphs

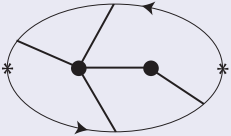
The ribbon graphs of a link diagram

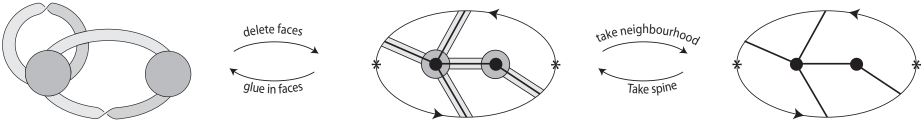
- Extend Tait graphs by associating a set of ribbon graphs to a link diagram (Dasbach, Futer, Kalfagianni, Lin & Stoltzfus ’08).
- Applications: Jones poly, HOMFLY-PT poly, Khovanov homology, knot Floer homology, Turaev genus, quasi-alternating links, the coloured Jones poly, signature, determinant, hyperbolic knots.

Ribbon graph



Cellularly embedded graph

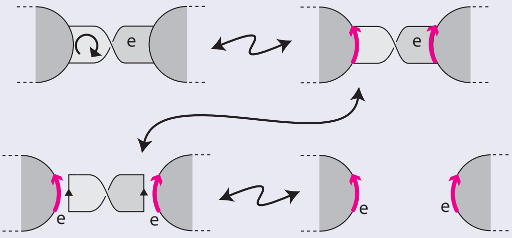




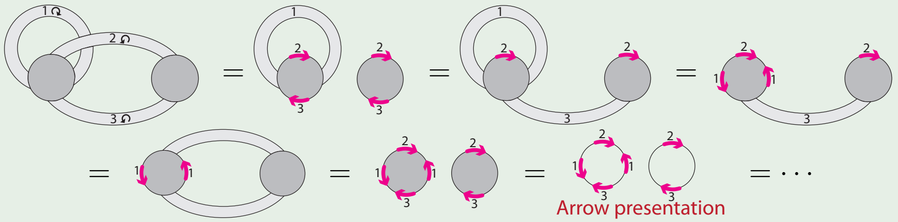
Arrow marked ribbon graphs

Edges can be described by pairs of coloured arrows on the boundary:

1. orient edge *e*
2. add arrows where *e* meets vertices
3. remove edge.

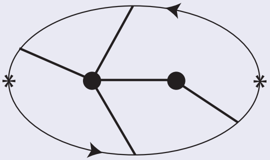


Example

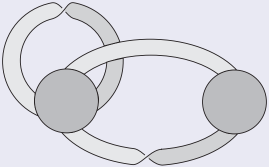


Embedded graphs

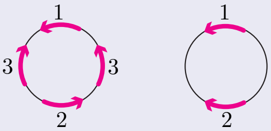
Cellularly embedded graph

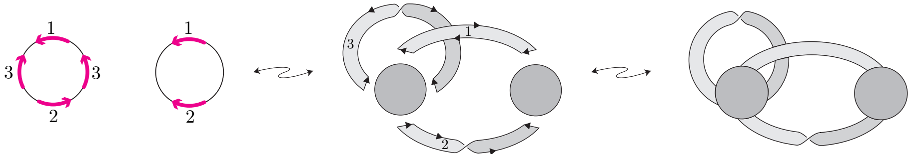


Ribbon graph



Arrow presentation

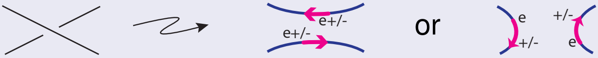




The ribbon graphs of a link diagram

The ribbon graphs of a link diagram (Dasbach, Futer, Kalfagianni, Lin & Stoltzfus ’08)

- Choose a decorated smoothing at each crossing:



- Gives an arrow presentation $\leftrightarrow$ ribbon graph.

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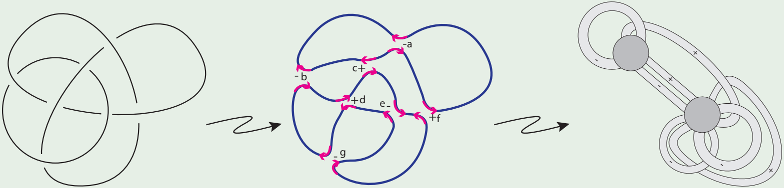

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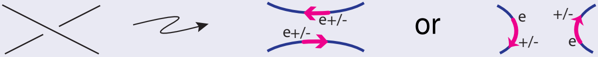

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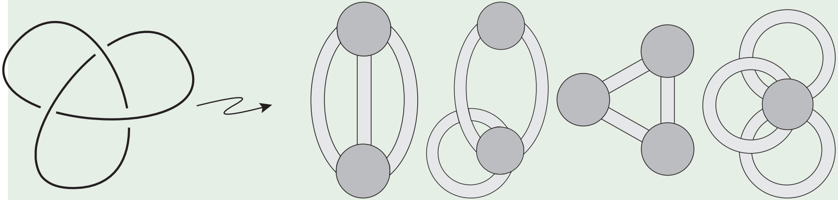
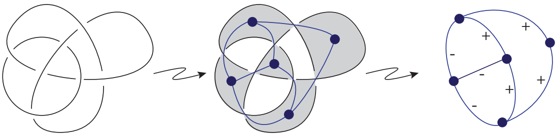
Example

The set of (unsigned) ribbon graphs of the trefoil is



Fundamental questions

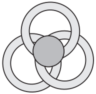
Recall Tait graphs: 

Tait graphs

- Tait graphs are duals.
- All plane graphs describe links.
- Tait graph describes a unique link diagram.

Ribbon graphs

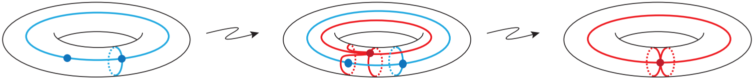
- Q1 How are RGs of a diagram related?
- Q2 Which RGs describe links?
- Q3 How are diagrams with same set of RGs related?

- Not all RG's describe links, e.g.  doesn't.
- Different link diagrams **can** give rise to the same set of RGs.

The geometric dual

The (geometric) dual G^* of a cellularly embedded graph G

- One vertex of G^* in each face of G.
- One edge of G^* whenever faces of G are adjacent.

$G =$  $= G^*$

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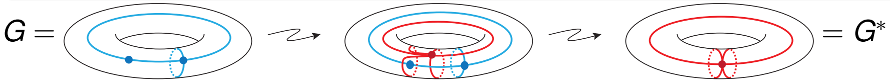
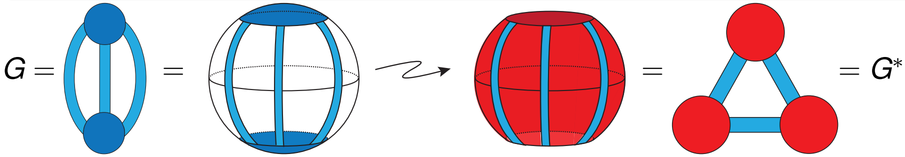


The (geometric) dual G^* of a ribbon graph G

- Fill in punctures of surface G with vertices of G^*,
- then delete vertices of G to get G^*.



- Note: markings on G induce markings on G^*.

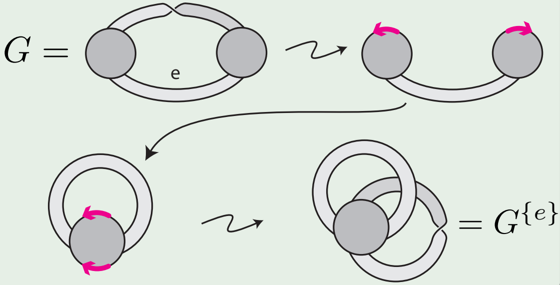
Partial duals

The **partial dual** G^A of G is obtained by forming the dual only at the edges in $A \subseteq E(G)$.

Definition: partial dual (Chmutov '09)

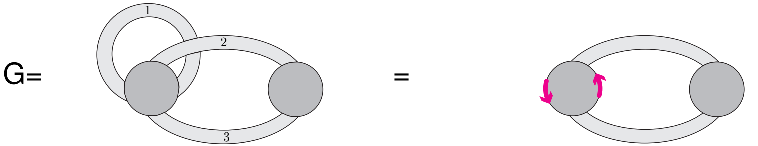
1. $A \subseteq E(G)$
2. Replace edges **not** in A by arrows.
3. Form geometric dual.
4. Add back edges.
5. Gives the **partial dual** G^A.

Example

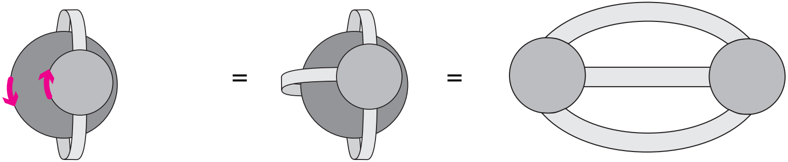


Another example

Forming G^A with $A = \{2, 3\}$.

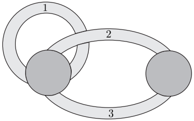


1: given G and A 2: "hide" edges not in A

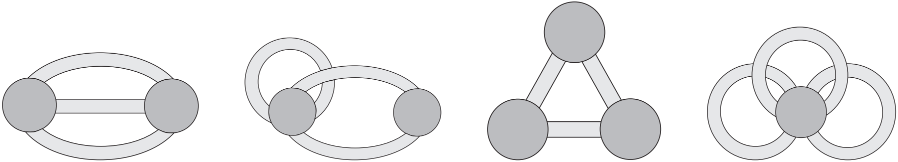


3: form the dual 4 & 5: add edge back to get G^A

The example continued...

G= 

has four partial duals (up to isomorphism):



- Observe that G and G^A can have very different graph theoretic and topological properties.

Relating the ribbon graphs of a link diagram

Q1: How the RGs of a link diagram are related

- Tait graphs are geometric duals.
- RGs are all partial duals.

In fact:

Proposition (Chmutov '09)

G is a RG of $D \iff G$ is a partial dual of a Tait graph of D.

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Reformulating the second question

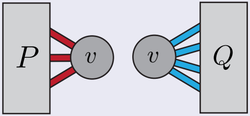
Q2: Which RGs describe links?
$\leftrightarrow$ Which RGs are partial duals of plane graphs?

plane-biseparations of ribbon graphs

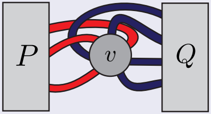
1-sums

$G = P \oplus Q$ if

- $G = P \cup Q$
- $P \cap Q = \{v\}$



Ribbon graphs P and Q.

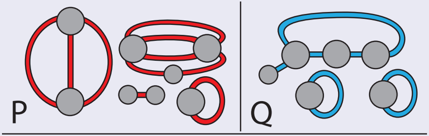


A 1-sum $P \oplus Q$

plane-biseparations

Idea:

- P, Q sets of plane RGs
- 1-sum elts. of P to elts. of Q



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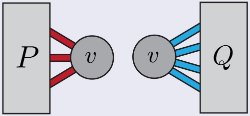

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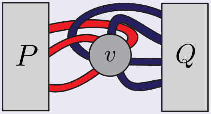

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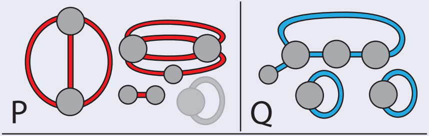

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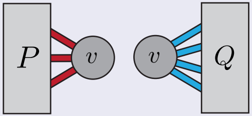

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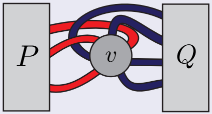

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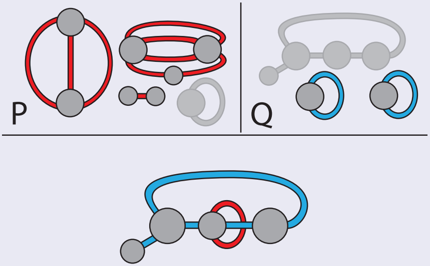

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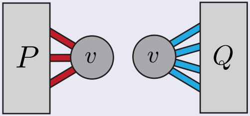

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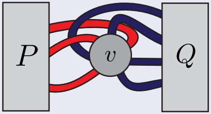

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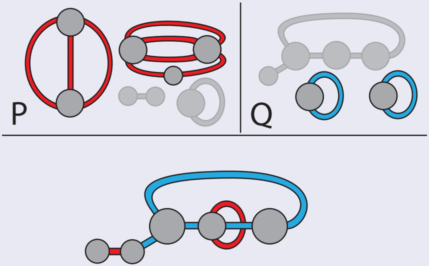

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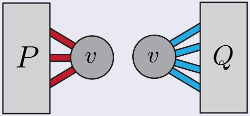

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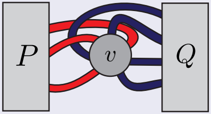

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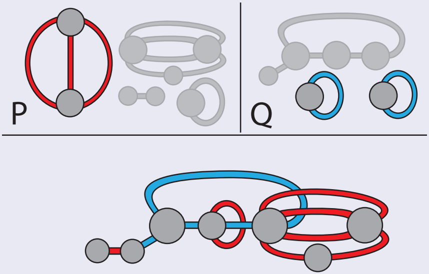

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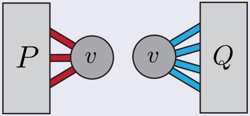

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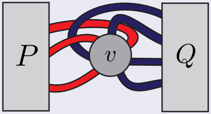

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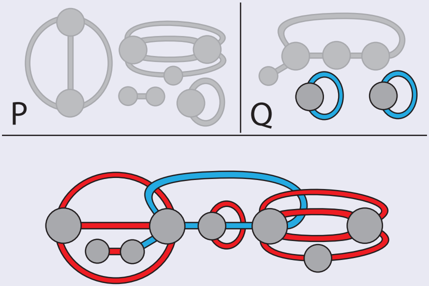

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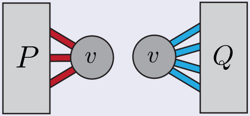

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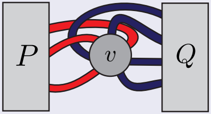

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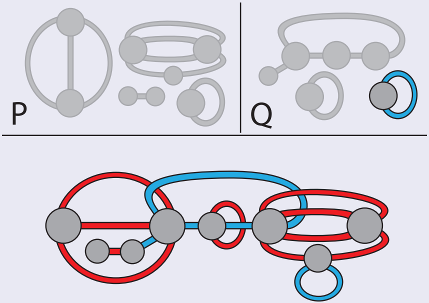

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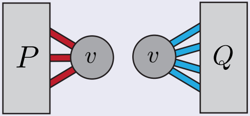

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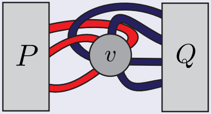

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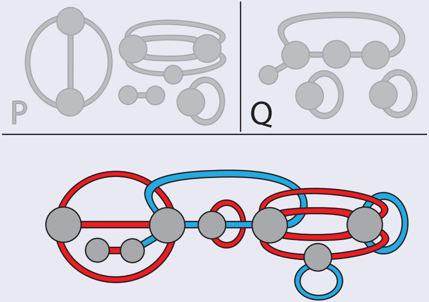

plane-biseparations of ribbon graphs

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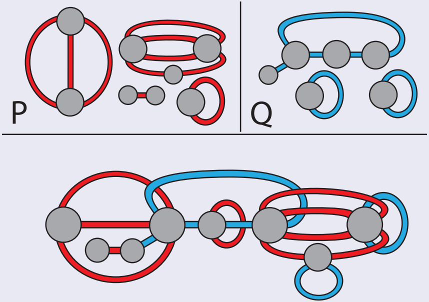


Definition

Formally: $A \subseteq E(G)$ defines a **plane-biseparation** if either

- $A = E(G)$ or $A = \emptyset$ and G plane; or
- G can be written as a sequence of 1-sums each of which involves a component of plane graphs $G|_A$ and $G|_{A^c}$.

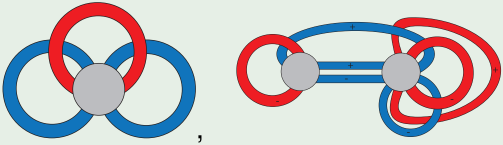
Characterizing plane partial duals

Plane-biseparations characterize partial duals of plane graphs:

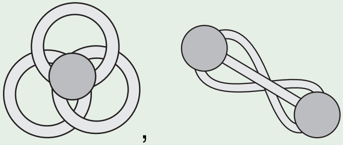
Theorem

Let G be a ribbon graph and $A \subseteq E(G)$. Then G^A is a plane ribbon graph if and only if A defines a plane-biseparation of G.

Example



partial duals of plane graphs



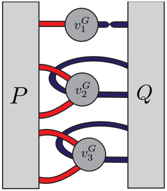
not p.ds of plane graphs

Q2: Which RGs describe links?

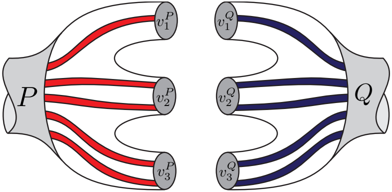
Corollary

G a ribbon graph of a link diagram $\iff$ it admits a plane-biseparation.

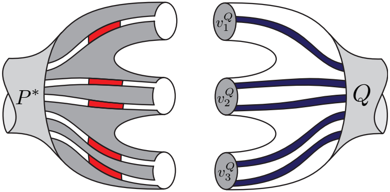
Idea of proof



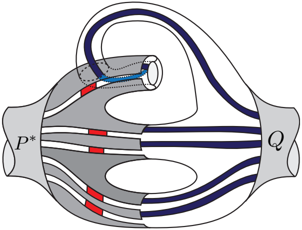
$G = P \oplus_3 Q$



$P \subset \Sigma_P$ and $Q \subset \Sigma_Q$



$P^* \subset \Sigma_P$ and $Q \subset \Sigma_Q$.



$(P \oplus_n Q)^{E(P)} \subset \Sigma$

Link diagrams presented by the same ribbon graphs

Tait graphs

- Tait graphs are duals.
- All plane graphs describe links.
- Tait graph describes a unique link diagram.

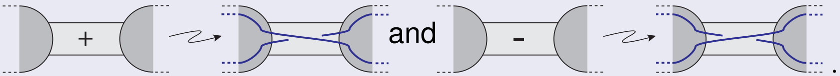
Ribbon graphs

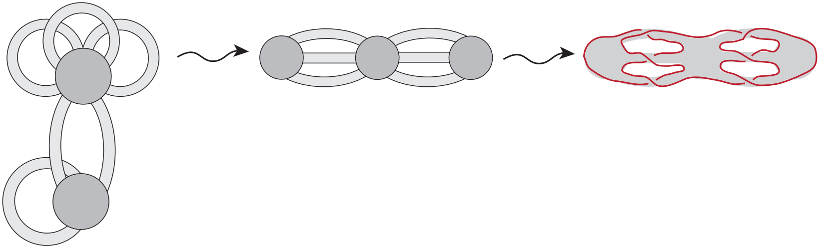
- Q1 How are RGs of a diagram related? ✓
- Q2 Which RGs describe links? ✓
- Q3 How are diagrams with same set of RGs related?

Link diagrams presented by the same ribbon graphs

Recovering link diagrams from a ribbon graph

- Given G form a plane partial dual G^A.
- Draw link on plane graph using





Link diagrams presented by the same ribbon graphs

Recovering link diagrams from a ribbon graph

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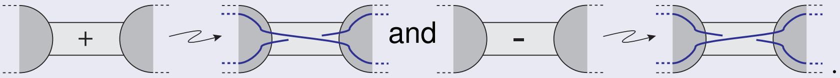
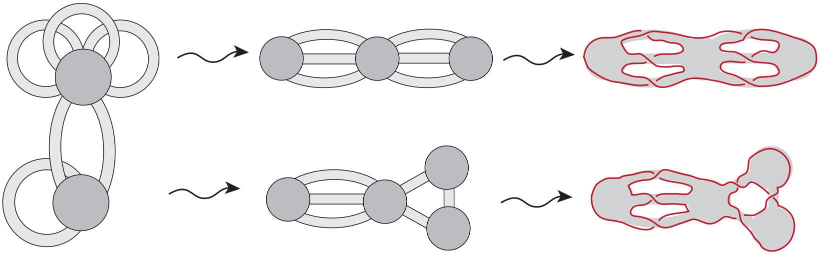


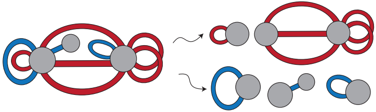


- How are the link diagrams related?

Relating link digrams

Approach

- Determine how plane partial duals are related.
- Look at how this affects link diagrams.

- G and G^A both plane
- $\implies$ A defines plane-biseparation of plane graph
- These have a special structure: 
- $\implies$ G and G^A related by 

- Embed in S^2 and look at link diagram.

Relating link digrams

- G and G^A both plane
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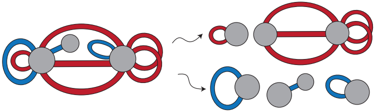
- These have a special structure: 
- $\implies$ G and G^A related by 

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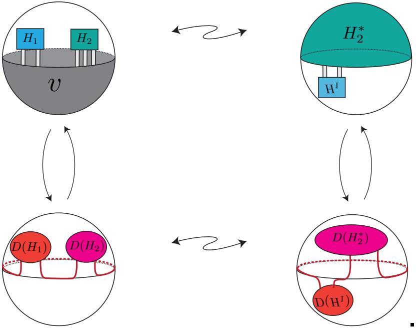
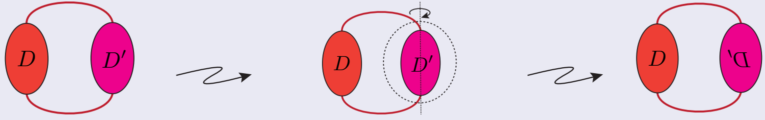


Relating link digrams

Theorem

D and D' represented by same set of ribbon graphs $\iff$ related by



$D_1 = D \# D'$ cut, flip and glue D_2

Corollary

Isotopy class of a link is represented by a unique set of ribbon graphs.

- Tait graphs are duals.
- All plane graphs describe links.
- Tait graph describes a unique link diagram.

- RGs of a diagram are partial duals
- RG describes a diagram $\iff$ admits a plane-biseparation.
- RGs describe diagrams up to “summand-flips”

Thanks!

References

- Mostly from: *Partial duals of plane graphs, separability and the graphs of knots*
- Bits from: *Separability and the genus of a partial dual*