The Shape of Space

Dr. Aditi Kar

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What is the shape of space?

Figure: Hubble 25th anniversary photo
Introduction

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Another *mathematical* aspect of the natural world: **shape**.
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Shape and symmetry: we use these to recognise objects around us.
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Another *mathematical* aspect of the natural world: **shape**.

Shape and symmetry: we use these to recognise objects around us.

Likewise, in advanced *geometry* and *group theory*, we classify objects via *shape*.
Some famous Geometers of today

Gromov, Abel Prize 2009.

Agol, Mathematics Breakthrough Prize 2016.

Mirzakhani, Fields Medal 2014.
Shape = local and global geometry of an object.
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Let’s start by talking Geometry.
Familiar geometry: here,
  - any two points are joined by a line,
  - 3 points define a triangle,
  - two lines either cross one another or are parallel.

Sum of Angles in a triangle = 180° or $\pi$. 
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Sum of Angles in a triangle $= 180^\circ$ or $\pi$.

No curvature

Flat geometry with $\kappa = 0$. 
Sphere

Sum of angles in a (geodesic) triangle $\geq 180^\circ$!

Positive curvature
Spherical geometry with $\kappa > 0$.

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**Positive curvature**

Spherical geometry with $\kappa > 0$. 

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Hyperbolic Geometry

Geometry of the saddle back

Triangles on the saddleback are very thin!
Euclid’s parallel postulate fails.

Negative Curvature
Hyperbolic geometry with \( \kappa < 0 \).
How different are these geometries?

In the plane, the area of a right triangle $= \frac{1}{2} \times \text{base} \times \text{height}$. 

[Diagram of a right triangle with labels for base and height]
Gauss Bonnet formula

Triangle on a sphere

Area = (sum of angles - $\pi$) $\times$ radius.
Gauss Bonnet formula

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Let’s check this.
Surface area of a sphere =
Triangle on a sphere

\[
\text{Area} = (\text{sum of angles} - \pi) \times \text{radius}.
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Surface area of a sphere = \(4\pi \times \text{radius}\).
Gauss Bonnet formula

Triangle on a sphere

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Let’s check this.
Surface area of a sphere = $4\pi \times$ radius.
And, there are 8 octants in the sphere.
Different shapes

Locally: all Euclidean. Globally: different!
Journey to $\infty$?

Are we ready...
Is the universe...

- bounded or infinite?
- Euclidean or flat ($\kappa = 0$), hyperbolic or saddleback ($\kappa < 0$) or, spherical ($\kappa > 0$)?
- Finally, how the universe is put together, i.e. simply connected space or not?
According to General Relativity, the shape of universe is determined by its density: the amount of mass spread over its volume.
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- **Too little mass**: **saddleback** geometry, infinite, with the universe expanding forever.

- **Too much mass**: spherical geometry, finite, expansion will stop and the universe will start to contract heading for the **BIG CRUSH**!

- **Just right**: Euclidean Geometry, infinite, the expansion will slow down gradually over an infinite amount of time.
The future will tell...

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Let the journey begin...
Let the journey begin...

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